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Critical exponents for square lattice trails with a fixed number of vertices of degree 4

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Abstract
We prove several previously conjectured results about the number of n-edge trails and n-edge embeddings of Eulerian graphs, each with a fixed number, k, of degree 4 vertices, in the lattice $\mathbb{Z}^2$. In particular, under the assumption that the relevant critical exponents exist, we prove that the difference between the critical exponent for closed trails (Eulerian graph embeddings) and that for self-avoiding circuits (polygons) is exactly k, the number of degree 4 vertices. Similarly, we prove that the difference between the critical exponent for either open trails or open Eulerian graph embeddings and that for self-avoiding walks is also k. These results are proved by establishing upper and lower bounds for the number of n-edge embeddings of closed (open) Eulerian graphs with k vertices of degree 4 in terms of the number of n-edge self-avoiding polygons (walks). The lower bounds are proved using a Kesten pattern theorem argument and the upper bounds are established by developing (based on a detailed case analysis) a method for removing vertices of degree 4 from an embedding by altering at most a constant (independent of n) number of vertices and edges of the embedding. The work presented here extends and improves the arguments first given in the work of Zhao and Lookman (1993 *J. Phys. A: Math. Gen.* **26** 1067–76).

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1. Introduction

The self-avoiding walk model has been established as a standard model of linear polymers in dilute solution in a good solvent. Similarly, lattice trail, lattice tree, lattice animal and self-avoiding polygon models have become standard models for branched and ring polymers. Recent studies of lattice trails have provided evidence that trails and self-avoiding walks are in the same universality class. Specifically, the evidence from exact enumeration and transfer matrix studies on a variety of lattices [1–7] indicates that the critical exponent for trails is the same as the critical exponent for self-avoiding walks. However, the exponential growth rate (with the number of steps or edges) of lattice trails is greater than that of self-avoiding walks [8]. There is an analogous situation with regard to lattice animals and lattice trees; that is, lattice animals and trees are believed to be in the same universality class, however, their exponential growth rates are different [9]. For this case, it is also known that c-animals (lattice animals with a fixed cyclomatic index c) have the same exponential growth rate as lattice trees and, based on combinatorial bounds, are expected to have a critical exponent which is increased by an amount c from that for lattice trees [10]. In this paper, a related question for lattice trails on the square lattice is investigated by focussing on the number of n-step trails with a fixed number of vertices of degree 4, and deriving combinatorial bounds relating the number of such trails to the number of n-step self-avoiding walks or polygons. The work presented here extends and improves the arguments first given by Zhao and Lookman [8] on a similar question. Since there is a connected subgraph of the lattice with either zero or two vertices of odd degree underlying each lattice trail, the approach used here is first to obtain bounds for the number of such lattice subgraphs. This is done by viewing the lattice subgraph as an embedding in the lattice of an abstract graph τ and using the known properties of $g_n(\tau)$, the number of such embeddings with n edges. In addition, new bounds are derived for $g_n(\tau)$ in the case that τ has either zero or two vertices of odd degree.

For $v = (v_1, \ldots, v_d)$, $w = (w_1, \ldots, w_d) \in \mathbb{R}^d$, define $\|v - w\| = |v_1 - w_1| + \cdots + |v_d - w_d|$. The hypercubic lattice $\mathbb{Z}^d$ will be viewed as the infinite graph with vertex set $V(\mathbb{Z}^d) = \{(x_1, \ldots, x_d) | x_i \in \mathbb{Z}, i = 1, \ldots, d\}$ and edge set $\mathcal{E}(\mathbb{Z}^d) = \{\{v, w\} | v, w \in V(\mathbb{Z}^d), \|v - w\| = 1\}$.

An n-step *self-avoiding walk* (n-SAW), ω, in the hypercubic lattice $\mathbb{Z}^d$ is a sequence of distinct vertices $r_0, r_1, \ldots, r_n$ in $V(\mathbb{Z}^d)$ such that r_{i-1} and r_i are joined by an edge in $\mathcal{E}(\mathbb{Z}^d)$ for $i = 1, \ldots, n$. The n-SAW ω is said to start at r_0 and end at r_n and, for $i = 1, \ldots, n$, the edge from r_{i-1} to r_i is called the ith step of the walk. The number of n-SAWs in $\mathbb{Z}^d$ starting at the origin is denoted by c_n^d.

An n-step *trail* (n-trail), σ, in $\mathbb{Z}^d$ starting at s_0 is a sequence of n distinct edges $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathcal{E}(\mathbb{Z}^d)$, such that $\alpha_i = \{s_{i-1}, s_i\}$ for $i = 1, \ldots, n$; α_i is said to be the ith step of the trail and is traversed from s_{i-1} to s_i, and vertex s_i is said to be traversed (or entered) from the ith step. Such a trail σ is also referred to as an n-trail connecting s_0 to s_n. The n-trail $\mathrm{rev}(\sigma) \equiv (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$, obtained by reversing the order of σ edges, is referred to as the *reverse trail* of σ. Note that if $s_0, s_1, \ldots, s_n$ are distinct vertices in $V(\mathbb{Z}^d)$, then σ is an n-SAW. The number of n-trails in $\mathbb{Z}^d$ starting at the origin is denoted by t_n^d. An n-step closed trail or *trailgon* is a trail such that $s_0 = s_n$. The number of n-trailgons in $\mathbb{Z}^d$ starting at the origin is denoted $\mathring{t}_n^d$. For any $i = 1, \ldots, n$, the n-trailgon $\mathrm{cyc}_{s_{i-1}}(\sigma) \equiv (\alpha_i, \alpha_{i+1}, \ldots, \alpha_n, \alpha_1, \alpha_2, \ldots, \alpha_{i-1})$, obtained from an n-trailgon σ by a cyclic permutation of its edges, is referred to as the *cyclic permutation of* σ starting at s_{i-1}. A trail which is not closed is called an *open trail*. The number of open n-trails in $\mathbb{Z}^d$ starting at the origin is denoted by $\check{t}_n^d$.

An n-step *self-avoiding circuit* (n-SAC) is a trailgon such that the vertices $s_0, \ldots, s_{n-1}$ are all distinct. The number of n-SACs in $\mathbb{Z}^d$ starting at the origin is denoted by q_n^d. For a given n-SAC σ, $\left\{\mathrm{cyc}_{s_{i-1}}(\sigma), \mathrm{cyc}_{s_{i-1}}(\mathrm{rev}(\sigma)), i = 1, \ldots, n\right\}$ forms a set of $2n$ distinct n-SACs originating from σ. This set of n-SACs can be regarded as a single geometrical entity, which is called an n-edge *self-avoiding polygon* (n-SAP). Equivalently an n-SAP is a connected n-edge, n-vertex subgraph of $\mathbb{Z}^d$ in which each vertex has degree 2. Two n-SAPs are considered equivalent if one is a translate of the other. The number of distinct n-SAPs in $\mathbb{Z}^d$ is denoted by p_n^d. Note that $q_n^d = 2np_n^d$.

An n-edge *lattice animal* (n-animal) in $\mathbb{Z}^d$ is any connected n-edge subgraph of $\mathbb{Z}^d$. Two n-animals are considered equivalent if one is a translate of the other. An abstract connected graph τ is said to be *homeomorphically irreducible* if it has no vertices of degree 2, or if it has exactly one vertex, and this vertex has degree 2 (i.e. a *loop graph*). Let $\mathcal{G}_{2d}$ be the set of all homeomorphically irreducible abstract connected graphs having at least one edge and with maximum vertex degree less than or equal to $2d$. An n-animal in $\mathbb{Z}^d$ is said to be an n-edge embedding of a graph $\tau \in \mathcal{G}_{2d}$ if the n-animal is *homeomorphic* to τ (i.e. isomorphic to τ when vertices of degree 2 are suppressed). An n-SAP is considered an embedding of the loop graph. An n-edge embedding of a graph $\tau \in \mathcal{G}_{2d}$ will be referred to as an n-tau. The number of distinct (up to translation) n-taus in $\mathbb{Z}^d$ is denoted by $g_n^d(\tau)$.

Hammersley [11, 12] proved (see also [13]) that

$$0 < \lim_{n\to\infty} n^{-1}\log c_n^d = \lim_{n\to\infty} n^{-1}\log p_n^d \equiv \log\mu_d < \log(2d-1) \tag{1.1}$$

where the second limit is taken through even values of n. Hence the number of walks and the number of polygons both increase with n, at the same exponential rate. Soteros *et al* [14] proved that for any $\tau \in \mathcal{G}_{2d}$ such that there exists $n > 0$ with $g_n^d(\tau) > 0$,

$$0 < \lim_{n\to\infty} n^{-1}\log g_n^d(\tau) = \log\mu_d \tag{1.2}$$

where the limit is taken through all values of n for which $g_n^d(\tau) > 0$. Hence the number of n-taus increases exponentially with n, at a rate which is independent of τ, and at the same rate as that for n-SAWs. Guttmann [2] proved the existence of the following limit for lattice trails, and Zhao and Lookman [8] proved the second inequality in

$$0 < \log\mu_d < \lim_{n\to\infty} n^{-1}\log t_n^d \equiv \log\mu_{T,d}. \tag{1.3}$$

It is believed that

$$\begin{aligned} c_n &= \mu^n n^{\gamma-1}\,\mathrm{e}^{o(\log n)} \\ p_n &= \mu^n n^{\alpha_{\text{sing}}-3}\,\mathrm{e}^{o(\log n)} = \mu^n n^{\gamma_0-1}\,\mathrm{e}^{o(\log n)} \\ q_n &= \mu^n n^{\alpha_{\text{sing}}-2}\,\mathrm{e}^{o(\log n)} = \mu^n n^{\gamma_c-1}\,\mathrm{e}^{o(\log n)} \\ g_n(\tau) &= \mu^n n^{\gamma_\tau-1}\,\mathrm{e}^{o(\log n)} \\ t_n &= \mu_T^n n^{\gamma-1}\,\mathrm{e}^{o(\log n)} \end{aligned} \tag{1.4}$$

where $\gamma_0 \equiv \alpha_{\text{sing}} - 2$ and $\gamma_c \equiv \gamma_0 + 1$, and the dependence on dimension, d, has been dropped for simplicity. In each case, it is also believed that the leading term in $\mathrm{e}^{o(\log n)}$ is a constant, independent of n, and that, for example, $\gamma > \gamma_0$. However, there is no rigorous proof for equations (1.4) beyond the existence of the *growth constants* μ and μ_T. The exponents γ, γ_0, γ_c, and γ_τ are known as *critical exponents* for SAWs, SAPs, SACs and n-taus, respectively. For fixed d and τ, these critical exponents are believed to be independent of the lattice structure. For the purposes of this paper, we assume that the limits defining these critical exponents do exist, that is, we assume equations (1.4) are true. Under this assumption, Guttmann and Whittington [15] (see also [8]) have proved that $\gamma_{\text{tadpole}} = \gamma_{\text{dumbbell}} = \gamma$. Furthermore, it has been conjectured that

$$\gamma_{\text{figure8}} \equiv \gamma_{\text{figure-eight}} = \gamma_0 + 1 \tag{1.5}$$

and

$$\gamma_{\text{twin-tailed tadpole}} = \gamma + 1 \tag{1.6}$$

but the best that has been proved is that $\gamma_0 + 1 \leqslant \gamma_{\text{figure8}} \leqslant \gamma_0 + 2$, and $\gamma + 1 \leqslant \gamma_{\text{twin-tailed tadpole}} \leqslant \gamma + 2$ ([8, 15, 16] and section 2 of this paper). In this paper, we focus on $\mathbb{Z}^2$ and establish some general critical exponent relationships for certain n-taus and specific subsets of n-trails. Then, for example, equations (1.5) and (1.6) follow from these general results.

In particular, let $\mathcal{G}_4^i(k)$ be the subset of $\mathcal{G}_4$ consisting of planar graphs with i vertices of odd degree and exactly k vertices of degree 4. For any $\tau \in \mathcal{G}_4^2(k)$ $\left(\mathcal{G}_4^0(k)\right)$, Euler's theorem [17] implies that τ contains an open (closed) trail which uses every edge of the graph, an *Euler trail*, and hence each n-tau may be converted to an open (closed) n-trail, by finding an Euler trail of the n-tau. We thus refer to $\tau \in \mathcal{G}_4^2(k)$ $\left(\mathcal{G}_4^0(k)\right)$ as an *open (closed) Eulerian graph with k vertices of degree 4*, or an *open (closed) k-graph*, for short. The number (up to translation) of n-edge embeddings in $\mathbb{Z}^2$ of all open k-graphs is defined by

$$\breve{E}_n(k) \equiv \sum_{\tau \in \mathcal{G}_4^2(k)} g_n(\tau) \tag{1.7}$$

while the number (up to translation) of n-edge embeddings in $\mathbb{Z}^2$ of all closed k-graphs is defined by

$$\overset{\circ}{E}_n(k) \equiv \sum_{\tau \in \mathcal{G}_4^0(k)} g_n(\tau). \tag{1.8}$$

Since for fixed k and i the number of graphs in $\mathcal{G}_4^i(k)$ is a positive constant independent of n, equations (1.2), (1.7) and (1.8) imply that for fixed k

$$0 < \lim_{n\to\infty} n^{-1} \log \breve{E}_n(k) = \lim_{n\to\infty} n^{-1} \log \overset{\circ}{E}_n(k) = \log \mu. \tag{1.9}$$

Consistent with the beliefs about c_n, one expects that for fixed k

$$\breve{E}_n(k) = \mu^n n^{\breve{\gamma}(k)-1}\, \mathrm{e}^{o(\log n)} \quad \text{and} \quad \overset{\circ}{E}_n(k) = \mu^n n^{\overset{\circ}{\gamma}(k)-1}\, \mathrm{e}^{o(\log n)}. \tag{1.10}$$

Then, since the conjectured equations (1.5) and (1.6) imply $\overset{\circ}{\gamma}(1) = \gamma_{\text{figure8}} = \gamma_0 + 1$ and $\breve{\gamma}(1) \geqslant \gamma_{\text{twin-tailed tadpole}} = \gamma + 1$, it has been conjectured that, for all $k \geqslant 0$

$$\overset{\circ}{\gamma}(k) = \gamma_0 + k \tag{1.11}$$

and

$$\breve{\gamma}(k) = \gamma + k. \tag{1.12}$$

Indeed, in this paper, we establish equations (1.11) and (1.12) by means of the following central result:

Main theorem. *There exist positive constants ϵ, $\tilde{C}$, $\tilde{D}$, C, D_0, D, N_ϵ and a set of graphs $\left\{\tau_k^i \in \mathcal{G}_4^i(k), k \geqslant 0, i \in \{0, 2\}\right\}$ such that for all $n \geqslant N_\epsilon$ and for any $k \geqslant 0$,*

$$\tilde{C}\binom{\lfloor \epsilon n\rfloor}{k} p_n \leqslant g_n\left(\tau_k^0\right) \leqslant \overset{\circ}{E}_n(k) \leqslant C^k \binom{2n}{k} p_n \tag{1.13}$$

and

$$\tilde{D}\binom{\lfloor \epsilon n\rfloor}{k} c_n \leqslant g_n\left(\tau_k^2\right) \leqslant \breve{E}_n(k) \leqslant D_0 (D)^k \binom{2n}{k} c_n \tag{1.14}$$

where the lower bounds hold for all $d \geqslant 2$ and the upper bounds hold for $d = 2$.

The graph τ_k^0 $\left(\tau_k^2\right)$ in the above theorem is referred to as the k-daisy (k-loop twin-tailed tadpole) graph and will be defined in detail later. However, the special cases τ_1^0, τ_1^2 and τ_0^0 are the figure-eight, twin-tailed tadpole and loop graphs, respectively, while τ_0^2 is the graph which is homeomorphic to an undirected SAW.

This main theorem implies for $d = 2$ that

$$g_n\left(\tau_k^0\right) = p_n n^k \,\mathrm{e}^{o(\log n)} \qquad \mathring{E}_n(k) = p_n n^k \,\mathrm{e}^{o(\log n)} \tag{1.15}$$

$$g_n\left(\tau_k^2\right) = c_n n^k \,\mathrm{e}^{o(\log n)} \qquad \breve{E}_n(k) = c_n n^k \,\mathrm{e}^{o(\log n)}. \tag{1.16}$$

Assuming equations (1.4), then for $d = 2$ and for any $k \geqslant 0$ equations (1.11) and (1.12) now follow, upon dividing equations (1.15) and (1.16) by μ^n, taking logarithms, dividing by $\log n$, and finally letting n go to infinity. Furthermore, the same argument yields

$$\gamma_{\tau_k^0} = \gamma_0 + k \quad \text{and} \quad \gamma_{\tau_k^2} = \gamma + k \tag{1.17}$$

for the k-daisy and k-loop twin-tailed tadpole graphs. For the case that $k = 1$ this gives equations (1.5) and (1.6), in $\mathbb{Z}^2$.

The main theorem also yields bounds for trails in terms of SAWs and SAPs from which further relationships between critical exponents can be obtained. To see this, consider $\breve{t}_n(k)$ ($\mathring{t}_n(k)$), the number of open (closed) n-trails in $\mathbb{Z}^2$ starting at the origin and with exactly k vertices of degree 4. Given any open (closed) k-graph $\tau \in \mathcal{G}_4^2(k)$ $\left(\mathcal{G}_4^0(k)\right)$, associated with each n-tau there is at least one distinct n-trail, σ. For the case of a closed k-graph, σ is one of a set of $2n$ distinct n-trailgons all obtained from σ by either a cyclic permutation of the edges of σ or by a cyclic permutation of the edges of rev(σ). Thus $2n\mathring{E}_n(k) \leqslant \mathring{t}_n(k)$. Meanwhile, for an open k-graph, σ and its reverse trail form a set of two distinct n-trails. Thus $2\breve{E}_n(k) \leqslant \breve{t}_n(k)$.

An upper bound on the number of distinct n-trails associated with an n-tau can be obtained by considering an upper bound on the number of ways to add edges (one at a time) from the n-tau to build an n-trail. For a closed k-graph, there are n ways to pick the first edge, α_1, of a trailgon and then two ways to pick one of its two endpoints as the trailgon's starting vertex, s_0. Every vertex of degree 2 in the n-tau is traversed once and every vertex of degree 4 is traversed twice by any n-trailgon of the n-tau. For any vertex, s_i, of degree 2 which is traversed from the ith step of a trailgon, there is only one way to pick the $(i + 1)$th step, α_{i+1}. For any vertex, s_i, of degree 4 which is traversed for the first time from the ith step of a trailgon, there are at most three ways to pick the edge α_{i+1} and then the next time the vertex is traversed (from step j say) there is only one way to pick the next edge, α_{j+1}. Thus $\mathring{t}_n(k) \leqslant 2(3)^k n\mathring{E}_n(k)$. For an open k-graph, there are two ways to pick an odd degree vertex as the trail's starting vertex, s_0. There are then at most three ways to choose α_1, the first edge of the trail, and should the trail ever traverse s_0, there is only one way to choose the next edge. As in the closed k-graph case, there is one way to choose the next edge when a trail traverses either a vertex of degree 2 or a vertex of degree 4 for the second time, and there are at most three ways to choose the next edge when a trail traverses a vertex of degree 4 for the first time. Finally, there are at most two ways to choose the next edge when a trail meets the last vertex for the first time. In summary, the following bounds are obtained

$$2\breve{E}_n(k) \leqslant \breve{t}_n(k) \leqslant 4(3)^{k+1}\breve{E}_n(k) \quad \text{and} \quad 2n\mathring{E}_n(k) \leqslant \mathring{t}_n(k) \leqslant 2(3)^k n\mathring{E}_n(k). \tag{1.18}$$

Thus, assuming equations (1.4), the bounds from equations (1.18), (1.13) and (1.14) imply that

$$\breve{t}_n(k) = \mu^n n^{\gamma+k-1}\,\mathrm{e}^{o(\log n)} \quad \text{and} \quad \mathring{t}_n(k) = \mu^n n^{\gamma_c+k-1}\,\mathrm{e}^{o(\log n)}. \tag{1.19}$$

The primary focus of the remainder of the paper will be the proof of the main theorem. However, we start by reviewing known results related to the main theorem bounds in section 2. In section 3, the Kesten pattern theorem is used to establish the main theorem lower bounds for all $d \geqslant 2$. In the remainder of the paper, the corresponding upper bounds for

$d = 2$ are established by mapping closed (open) k-graph embeddings to SAPs (SAWs), in such a way that the original embedding is affected only locally around the vertices of degree 1, 3 or 4, while the rest of the graph remains untouched. To construct the proper maps, an algorithm is developed which allows for the removal (one at a time) of each vertex of degree 4 in lexicographical order. This approach is broken down into several stages, as follows: removal of an isolated vertex of degree 4, from an n-edge embedding of a closed k-graph, within a 4×4 square box (sections 4.2.1 and 4.2.2); removal of the top vertex of degree 4 in an n-edge embedding of a closed k-graph, within a 10×10 box (section 5); removal of the top vertex of degree 4 in an n-edge embedding of an open k-graph, within a 12×12 box (section 6); an algorithm to remove all vertices of degree 4, by repeatedly removing consecutive top vertices of degree 4 (section 7). Using combinatorial arguments, the main theorem upper bound is at last established in section 8.

2. Review of existing bounds for $g_n(\tau)$

We begin by reviewing results previously known with regard to $g_n(\tau)$ for specific choices of τ: namely, tadpoles, figure-eights and twin-tailed tadpoles.

The unit vectors in $\mathbb{Z}^d$ are denoted by the d-tuples $u_1 = (1, 0, \ldots, 0)$, $u_2 = (0, 1, 0, \ldots, 0), \ldots$, $u_d = (0, \ldots, 0, 1)$. Vertices in $\mathbb{Z}^d$ are ordered lexicographically according to their coordinates. Based on this ordering, the *top vertex* (*bottom vertex*) of an n-animal, ω, is defined to be the largest (smallest) vertex amongst the vertices of ω.

For the tadpole graph, an upper bound for $g_n(\text{tadpole})$ in terms of c_n can be obtained by considering any n-tadpole whose vertex of degree 1 is located at the origin. Deleting an appropriate edge (there are always two choices for this edge) adjacent to the vertex of degree 3 in the n-tadpole creates an $(n-1)$-SAW starting at the origin. Since there are at most $(2d-1)$ ways to add a step to an $(n-1)$-SAW to create an n-tadpole, this argument yields the bound $2g_n(\text{tadpole}) \leqslant (2d-1)c_{n-1}$. Zhao and Lookman [8] have described a method for obtaining a lower bound for $g_n(\text{tadpole})$ in terms of c_n by converting an n-SAW into an $(n+i)$-tadpole for either $i = 1, 2$ or 3 (note that it is possible that up to two n-SAWs could yield the same n-tadpole by this construction). These arguments apply to any hypercubic lattice, $\mathbb{Z}^d$, and combined with the fact that $c^d_{n-1} \leqslant c^d_n$ [18] yield the following.

Theorem 1 (Zhao and Lookman [8]). *For any $d \geqslant 2$ and any $n \geqslant 2$*

$$c^d_n \leqslant 2\sum_{i=1}^{3} g^d_{n+i}(\text{tadpole}) \leqslant 3(2d-1)c^d_{n+2} \tag{2.1}$$

and hence, assuming the existence of the limit that defines γ in equation (1.4), $\gamma(\text{tadpole}) = \gamma$.

For the figure-eight graph, Guttmann and Whittington [15] developed a useful one-to-one mapping between n-edge embeddings of the figure-eight graph (n-figure8) in $\mathbb{Z}^2$ and a subset of doubly vertex-rooted polygons in $\mathbb{Z}^2$. The map is based on the basic moves shown in figure 1.

The transformation used to go from the figure-eight embedding depicted in figure 1(a) (or its reflection through the vertical axis) to the 'solution' configuration is referred to as a *flip* and the inverse transformation as a *reverse flip*. Note that the result of a flip on an n-figure8 can be either an n-SAP or another n-figure8. The transformation used to go from the figure-eight embedding depicted in figure 1(b) (or its reflection through either the line $y = -x$ or the vertical axis or both) to the 'solution' configuration is referred to as a *U-turn reduction* and

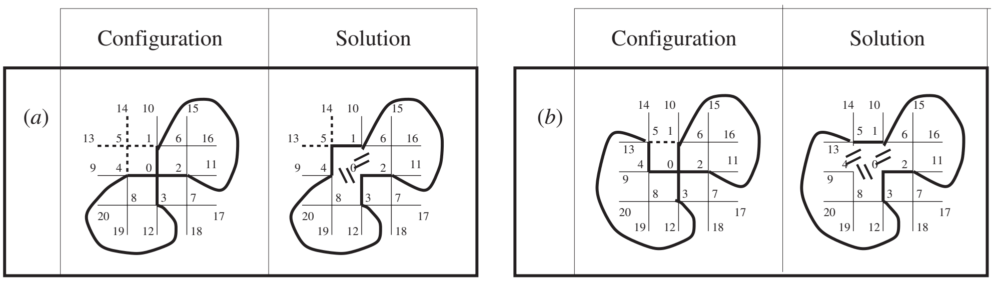

Figure 1. (*a*) depicts a flip transformation and (*b*) depicts a U-turn transformation. These figures also show solutions for cases (1) and (2) in section 4.2.2. Bold solid edges are occupied, dashed edges are not occupied, and fine solid edges may or may not be occupied. Double hash marks indicate edges which have been removed, during the solution.

the inverse transformation as a *U-turn expansion*. A U-turn reduction always converts an n-figure8 into an n-SAP.

Theorem 2 (Guttmann and Whittington [15]). *There exists a one-to-one map* Φ *such that for any n-figure8,* ω*, in* $\mathbb{Z}^2$*,*

$$\Phi : \omega \to (\omega', \psi_1, r) \tag{2.2}$$

where ω' *is an m-edge polygon,* ψ_1 *is a distinct vertex of* ω' *(corresponding to the vertex of degree 4 in* ω*),* $r < n$ *is a positive integer, and* (ω', ψ_1, r) *has the property that for a unique choice of* $\delta \in \{-1, 1\}$ *the following holds. (Note that* $\delta = 1$ (-1) *corresponds to a class A (B) n-figure8* ω *as defined in [15].)*

The edges $\{\psi_1 - \delta u_1, \psi_1\}$ *and* $\{\psi_1 + u_2, \psi_1\}$ *are not edges of* ω' *and either (i) exactly one of the edges* $\{\psi_1 - \delta(r-1)u_1 + ru_2, \psi_1 - \delta ru_1 + ru_2\}$ *or* $\{\psi_1 - \delta ru_1 + (r-1)u_2, \psi_1 - \delta ru_1 + ru_2\}$ *is an edge of* ω'*, or (ii) both of the edges* $\{\psi_1 - \delta(r-1)u_1 + ru_2, \psi_1 - \delta ru_1 + ru_2\}$ *and* $\{\psi_1 - \delta ru_1 + (r-1)u_2, \psi_1 - \delta ru_1 + ru_2\}$ *are edges of* ω'*.*
Furthermore, in case (i) $m = n - 2$ *and performing (starting at* ψ_1*) a sequence of r reverse flips followed by a U-turn expansion turns* ω' *into* ω*; and in case (ii),* $m = n$ *and performing a sequence of r reverse flips starting at* ψ_1 *turns* ω' *into* ω*.*

Corollary 1. *There exist positive constants A and* N_A *such that for all even* $n \geqslant N_A$

$$Anp_n \leqslant g_n(\text{figure8}) \leqslant n^2 p_n. \tag{2.3}$$

Proof. The lower bound in equation (2.3) was proved in Zhao and Lookman [8] by a Kesten pattern theorem argument (see section 3 of this paper for more details).

The upper bound in equation (2.3) comes from determining an upper bound on the number of distinct ω which yield the same ω' under the map Φ of theorem 2. Madras [19] pointed out that (contrary to the claim in [15] that only ψ_1 is needed) it is essential to know both ψ_1 and r to determine ω from ω' completely. For example, if only ω' and ψ_1 are known, then ω' (rooted at ψ_1) at the top of figure 2 could have resulted from any of the four ω shown below it in figure 2. In particular, for any triple $\Phi(\omega) = (\omega', \psi_1, r)$ there are r (one for each value of r', $1 \leqslant r' \leqslant r$) distinct figure-eight embeddings that yield triples of the form (ω', ψ_1, r'). There are no more than n choices for r and no more than n choices for ψ_1 so that the upper bound in equation (2.3) is an upper bound on the number of distinct triples (ω', ψ_1, r') that could have resulted from distinct figure-eight embeddings. □

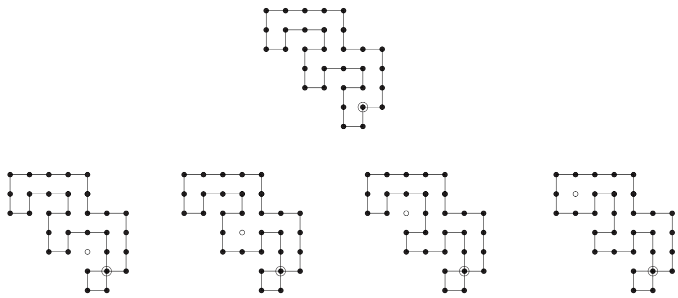

Figure 2. The topmost rooted polygon, a theorem 2 case (ii) ($\delta = 1$) polygon, could have resulted from any one of the four figure-eight embeddings shown below it for $r = 1, 2, 3, 4$, from left to right respectively.

Gaunt *et al* [16] showed that the same transformation could be applied to twin-tailed tadpoles in $\mathbb{Z}^2$ to produce a self-avoiding walk.

Theorem 3 (Gaunt *et al* [16])**.** *The map Φ of theorem 2 acts on an n-twin-tailed tadpole, ω, in $\mathbb{Z}^2$ such that*

$$\Phi : \omega \to (\omega', \psi_1, r) \tag{2.4}$$

where ω' is an m-edge undirected walk with $n-2 \leqslant m \leqslant n$, ψ_1 is a vertex of ω' (corresponding to the vertex of degree 4 in ω), $r < n$ is a positive integer and exactly one of the conditions of theorem 2 holds. Furthermore, ω' can be turned into ω as described in theorem 2 with the additional possibility that $m = n - 1$ in case (i).

Corollary 2. *There exist positive constants B and N_B such that for all $n \geqslant N_B$*

$$Bnc_n \leqslant g_n(\text{twin-tailed tadpole}) \leqslant n^2 c_n. \tag{2.5}$$

Zhao and Lookman [8] also discussed how some of these arguments can be extended to k-loop twin-tailed tadpole graphs (see the next section for a definition of these graphs) for fixed k. However, their approach does not lead to upper bounds in the form of those in equation (1.14).

3. Lower bound

The goal of this section is to establish the lower bounds in equations (1.13) and (1.14) for $\mathbb{Z}^d$. In order to define the graphs τ_k^i, $i = 0, 2$, that appear in these lower bounds, it is useful to first introduce some consequences of Kesten's pattern theorem [20] and then show how they can be used to prove that all except exponentially few SAWs or SAPs can be converted to embeddings of an appropriate k-graph.

A *Kesten pattern* [20] is any prescribed finite-step SAW P such that there exists at least one longer SAW ω, which contains three or more translates of P. Given a Kesten pattern P, define $c_n(\epsilon n, P)$ to be the total number of n-SAWs starting at the origin which contain more than $\lfloor \epsilon n \rfloor$ translates of the Kesten pattern P. The following is a consequence of Kesten's pattern theorem [20], which holds for any $d \geqslant 2$:

Theorem 4. *Given any Kesten pattern P, $\exists\, \epsilon_P > 0$ such that*

$$\lim_{n\to\infty} \frac{c_n(\epsilon_P n, P)}{c_n} = 1. \tag{3.1}$$

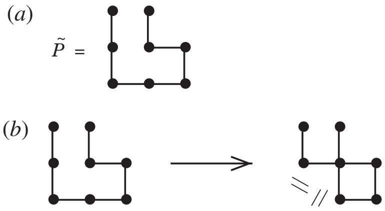

Figure 3. (*a*) An undirected Kesten pattern $\tilde{P}$. (*b*) *L*-to-Loop transformation of $\tilde{P}$ into a loop shape, via removal of two edges and addition of two other edges.

Corollary 3. *Given any Kesten pattern P, $\exists \epsilon_P > 0$ and $N_P > 0$ such that $\forall n \geqslant N_P$,*

$$\frac{c_n}{2} < c_n(\epsilon_P n, P) \leqslant c_n. \tag{3.2}$$

Let $\tilde{P}$ be the undirected walk (obtained by ignoring the directions on the edges) associated with the Kesten pattern P. Define $p_n(\epsilon n, P)$ to be the total number, up to translation, of n-SAPs such that more than $\lfloor \epsilon n \rfloor$ distinct translates of $\tilde{P}$ appear as subgraphs of the SAP. Then

Corollary 4 (Sumners and Whittington [21])**.** *Given any Kesten pattern P, $\exists \epsilon_P > 0$ such that*

$$\lim_{n\to\infty} \frac{p_n(\epsilon_P n, P)}{p_n} = 1. \tag{3.3}$$

Corollary 5. *Given any Kesten pattern P, $\exists \epsilon_P > 0$ and $M_P > 0$ such that $\forall n \geqslant M_P$,*

$$\frac{p_n}{2} < p_n(\epsilon_P n, P) \leqslant p_n. \tag{3.4}$$

Let $\tilde{P}$ denote the undirected walk shown in figure 3(*a*) and let P be a fixed Kesten pattern obtained by assigning a direction to $\tilde{P}$. Let $\epsilon_P > 0$ and $M_P > 0$ ($N_P > 0$) be as required for the result in corollary 5 (corollary 3). Given any n-SAP (SAW) containing $M \geqslant \lfloor \epsilon_P n \rfloor \geqslant k \geqslant 1$ copies of $\tilde{P}$ (P), the n-SAP (SAW) can be converted to an n-embedding of the graph τ_k^0 $\left(\tau_k^2\right)$, referred to here as a k-daisy (k-loop twin-tailed tadpole) graph, by performing k L-to-loop transformations, defined in figure 3(*b*), and then ignoring the orientation of the edges (if necessary). Note that for a given k, this conversion process uniquely defines the graph τ_k^0 $\left(\tau_k^2\right)$. There are at least $\binom{M}{k} \geqslant \binom{\lfloor \epsilon_P n \rfloor}{k}$ ways to perform k L-to-loop transformations. From this and the lower bounds from equations (3.2) and (3.4), we have

$$\frac{1}{2}\binom{\lfloor \epsilon_P n \rfloor}{k} c_n \leqslant g_n\left(\tau_k^2\right) \leqslant \sum_{\tau\in\mathcal{G}_4^2(k)} g_n(\tau) = \breve{E}_n(k) \tag{3.5}$$

for all $n \geqslant N_P$ (a bound of this form was first derived by Zhao and Lookman [8]), and

$$\frac{1}{2}\binom{\lfloor \epsilon_P n \rfloor}{k} p_n \leqslant g_n\left(\tau_k^0\right) \leqslant \sum_{\tau\in\mathcal{G}_4^0(k)} g_n(\tau) = \mathring{E}_n(k) \tag{3.6}$$

for all $n \geqslant M_P$. Note that for equation (3.5), the fact that $g_n\left(\tau_0^2\right) = c_n/2$ was used for the case $k = 0$. Thus, the lower bounds of equations (1.13) and (1.14) are proved, with $\tilde{C} = \tilde{D} = \frac{1}{2}$.

One consequence of the above lower bounds is that it can be shown (contrary to the conjecture in [16]) that the critical exponent for a graph with $b > 0$ cut-edges is not necessarily $\gamma + b - 1$. Indeed, a graph, $\hat{\tau}$, whose critical exponent (if it exists) is strictly greater than

$\gamma + b - 1$ can be constructed as follows. Let ω be an n-edge embedding in $\mathbb{Z}^d$ of τ_k^0 (the k-daisy graph) for some $k \geqslant 1$. Consider a self-avoiding circuit of ω which intersects exactly one degree 4 vertex, v, of ω, i.e. consider one of the 'petals' of the k-daisy graph. Remove an edge of the SAC incident on v. The resulting $(n-1)$-edge embedding, ω', is homeomorphic to a graph $\hat{\tau} \in \mathcal{G}_4^2(k-1)$, referred to as the $(k-1)$-daisy tadpole. This argument and equation (3.6) lead to

$$\binom{\lfloor \epsilon_P n \rfloor}{k} p_n \leqslant g_n\left(\tau_k^0\right) \leqslant g_{n-1}(\hat{\tau}). \tag{3.7}$$

Assuming equations (1.4) hold with $\gamma > \gamma_\circ$, let $\delta = \gamma - \gamma_\circ > 0$. Then for $k \geqslant \delta + 1$, dividing equation (3.7) by μ^n, taking logarithms, dividing by $\log n$, and finally letting n go to infinity leads to

$$\gamma_{\hat{\tau}} \geqslant \gamma_\circ + k = \gamma + k - \delta \geqslant \gamma + 1 = \gamma + b > \gamma + b - 1 \tag{3.8}$$

where $b = 1$ is the number of cut-edges in $\hat{\tau}$.

4. Isolated vertex of degree 4

4.1. Preliminary definitions

For this discussion, let $G = (V_G, \mathcal{E}_G)$ be a graph consisting of a set of vertices, V_G, and their incident edges, $\mathcal{E}_G$. For all of the graphs, G, which follow, we impose the conditions that each vertex $v \in V_G$ has at least one incident edge and that each edge $\alpha \in \mathcal{E}_G$ has two distinct vertices, say $v \in V_G$ and $w \in V_G$, incident on it. Hence G is completely defined by $\mathcal{E}_G$ and, where the context is clear, $\mathcal{E}_G$ is used interchangeably with G. Because of these stipulations, it is henceforth to be understood that, if G is a subgraph of H (denoted by $G \subset H$) then the graph $H \backslash G$ (or $H \backslash \mathcal{E}_G$) is the subgraph of H whose edge set is $\mathcal{E}_H \backslash \mathcal{E}_G$ and whose vertex set consists of precisely those vertices in V_H which are incident on edges in $\mathcal{E}_H \backslash \mathcal{E}_G$. The *degree* of a vertex v in a graph G, the number of distinct edges of G incident on v, is denoted by $\deg_G(v)$. When the context is clear, we drop the subscript G and write $\deg(v)$. Further, if $\deg_G(v) = m$ is odd (even), then we say that v is an *odd vertex* (*even vertex*). In the case that G is a subgraph of $\mathbb{Z}^2$, for any $v \in \mathbb{Z}^2$ such that $v \notin V_G$ we define $\deg_G(v) = 0$.

A trail, p, of G is a sequence of distinct edges $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathcal{E}_G$, such that $\alpha_i = \{s_{i-1}, s_i\}$ for $i = 1, \ldots, n$. If there exists $1 \leqslant i \leqslant j \leqslant n$ such that $p' = (\alpha_i, \alpha_{i+1}, \ldots, \alpha_j)$, then p' is said to be a *consecutive subtrail* of p.

Let G be a graph. Two vertices $v \in V_G$ and $w \in V_G$ are said to be *nearest neighbours (NN)*, if there exists an edge in $\mathcal{E}_G$, connecting v and w. For a connected graph G, vertex v is a *cut vertex* if G becomes disconnected after the removal of v and all of its incident edges.

Next, we highlight some properties of closed (open) Eulerian graphs, those graphs which contain a trail that uses each edge of the graph.

Theorem 5 (Euler's theorem [17])**.** *A connected graph contains a closed (open) Euler trail if and only if all the vertices of the graph are even (exactly two vertices are odd).*

The following results are consequences of Euler's theorem and its proof.

Lemma 1. *Let G be a connected graph with only even vertices. Let $a \in V_G$ be a vertex with* $\deg_G(a) = 4$. *Let $S = \{b, c, d, h\}$ be the set of four NN vertices to a. Choose any one of the vertices in S, say c. Then G has a trail, p, not containing a, from c to some vertex $v \in S \backslash \{c\}$.*

We need to make two more lemmas (further consequences of Euler's theorem) which will determine whether a connected graph remains connected, upon the deletion of a certain choice

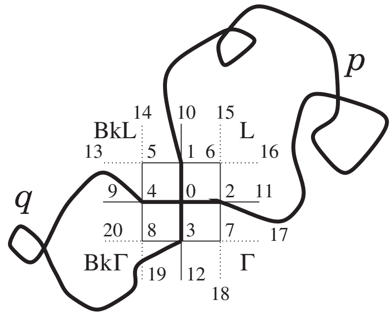

Figure 4. Isolated vertex of degree 4 in Eulerian trail E: $\deg_E([0]) = 4$ and $\deg_E([i]) \in \{0, 2\}$, for $1 \leqslant i \leqslant 20$. The bold solid curve indicates the edges which are occupied. All other edges may or may not be occupied. The trail p connects vertex [1] to vertex [2]; q connects [3] to [4]; neither p nor q contains [0]; and p and q may share no common edge. L, Γ, BkL, and BkΓ are indicated by the dotted lines.

of edges. The first lemma involves the removal of a subtrail. The second lemma involves classifying a vertex of degree 4 as cut or not cut. For the remainder of this section, we assume that $\tau \in \cup_{k\geqslant 0}\mathcal{G}_4^0(k)$, the set of closed k-graphs, for $k \geqslant 0$.

Lemma 2. *Let G be any graph which is homeomorphic to τ. Suppose that for some $j \geqslant 2$, we are given a sequence of distinct edges, $p = (\alpha_1, \alpha_2, \ldots, \alpha_j)$, which forms an open (closed) trail of G and let $E = \{\alpha_1, \ldots, \alpha_j\}$. If there exists a closed Euler trail of G which contains p as a consecutive subtrail then the graph $G\backslash E$ contains an open (closed) Euler trail and hence is connected.*

Recall that if G is an n-tau, then G is an n-edge connected subgraph of $\mathbb{Z}^2$ which is homeomorphic to τ and hence G is planar and has only even vertices. More generally, we state that G is an n-edge *plane embedding* of τ if G is a planar depiction in $\mathbb{R}^2$ of an n-edge graph which is homeomorphic to τ. Thus every n-tau is also an n-edge plane embedding of τ.

Lemma 3. *Let G be a plane embedding of τ. Let $\deg_G(a) = 4$, for some vertex $a \in V_G$. Let c, d, b and h be the four NN vertices of a, listed in clockwise order around a. Then a is not a cut vertex if and only if both of the following occur:*

(i) G has a trail r, not containing a, which connects c and b;
(ii) G has a trail s, not containing a, which connects d and h.

To facilitate further discussion, we introduce $S_{[0]}$, a subgraph of $\mathbb{Z}^2$, consisting of 21 vertices in $V(\mathbb{Z}^2)$, labelled $0, 1, \ldots, 20$, and 24 corresponding incident edges in $\mathcal{E}(\mathbb{Z}^2)$, shown in figure 4. $S_{[0]}$ is basically a 4×4 square grid, minus the four outer corners. The vertex labelled as i is referred to as *vertex* $[i]$, or simply $[i]$, for $i = 0, 1, 2, \ldots, 20$. In addition, we use $[i, j]$ to denote the edge with endpoints $[i]$ and $[j]$. When we wish to indicate direction along an edge, we use $\overrightarrow{[i, j]}$ to denote the traversal of the edge in the direction from vertex $[i]$ to $[j]$. We also observe that the line $y = x$ goes through the vertices [0], [6] and [8]. For simplicity, the edge set, $\{e_1, \ldots, e_m\} = \cup_{i=1}^m \{e_i\}$, which defines a subgraph of $S_{[0]}$ will be denoted by $\cup_{i=1}^m e_i$.

The boundary SAP of $S_{[0]}$ defined by the vertex set, {[11], [17], [7], [18], [12], [19], [8], [20], [9], [13], [5], [14], [10], [15], [6], [16]}, divides the plane $\mathbb{R}^2$ into a bounded region, referred to as $I(S_{[0]})$ or *inside* $S_{[0]}$, and an unbounded region, referred to as $O(S_{[0]})$ or *outside* $S_{[0]}$. It is assumed that the boundary SAP is part of $I(S_{[0]})$.

For convenience, we set $L = [6, 15] \cup [6, 16]$ and $\Gamma = [7, 17] \cup [7, 18]$ to be the two L-shaped subgraphs in the upper and lower right-hand corners of the grid, respectively. The 'backward' L-shapes in the upper and lower left-hand corners of the grid are similarly defined as $\mathrm{BkL} = [5, 13] \cup [5, 14]$ and $\mathrm{Bk}\Gamma = [8, 19] \cup [8, 20]$ (see figure 4).

Suppose G is any subgraph of $\mathbb{Z}^2$ with at least one vertex of degree 4. Let v be a vertex of degree 4 in G. Translate the 4×4 square grid $S_{[0]}$ so that vertex [0] in $S_{[0]}$ coincides with vertex v in G. If $\deg_G([i]) \in \{0, 2\}$, for $1 \leqslant i \leqslant 20$, then we say that v is an *isolated vertex of degree 4* in G.

4.2. Isolated vertex lemma

Lemma 4. *Given any closed k-graph τ, let G be an n-tau in $\mathbb{Z}^2$. Suppose v is any isolated vertex of degree 4 in G, and translate the* 4×4 *box* $S_{[0]}$ *so that* [0] *coincides with v. Then it is possible, by only altering edges and vertices of G within* $S_{[0]}$*, to construct a new closed Eulerian m-animal G′ rooted at* [0]*, with* $k-1$ *vertices of degree 4, and with* $m \in \{n, n-2, n-4\}$*, such that* $G' = G$ *outside* $S_{[0]}$*, and* $\deg_{G'}([0]) = 2$.

Furthermore, this can be accomplished so that if we are given an open trail $\mathcal{C}$ of G consisting of edges outside $S_{[0]}$ and such that $\mathcal{C}$ is a consecutive subtrail of an Euler trail of G, then G′ also has an Euler trail which contains $\mathcal{C}$ as a consecutive subtrail.

Proof. The purpose of the second paragraph of the lemma is so that this result can be later used to deal with the case where τ is an open k-graph. Also for this purpose, we note that the proof presented next in fact applies to any n-edge plane embedding, G, of τ such that $G \cap I(S_{[0]})$ is a subgraph of $S_{[0]}$.

To begin the proof, let G, v, $S_{[0]}$ and $\mathcal{C}$ be as described in the statement of the lemma. Euler's theorem and the planarity of G dictate that there is an Euler trail E of G in at least one of the following three forms:

(I)

$$E = \overrightarrow{[0, 1]}; r; \overrightarrow{[3, 0]}; \overrightarrow{[0, 2]}; s; \overrightarrow{[4, 0]} \tag{4.1}$$

where r and s are subtrails of E, neither r nor s contains [0], and either $\mathcal{C}$ or $\mathrm{rev}(\mathcal{C})$ is a consecutive subtrail of one of r or s.

(IIa)

$$E = \overrightarrow{[0, 1]}; p; \overrightarrow{[2, 0]}; \overrightarrow{[0, 3]}; q; \overrightarrow{[4, 0]} \tag{4.2}$$

where p and q are subtrails of E, neither p nor q contains [0], and either $\mathcal{C}$ or $\mathrm{rev}(\mathcal{C})$ is a consecutive subtrail of one of p or q.

(IIb)

$$E = \overrightarrow{[0, 1]}; p; \overrightarrow{[4, 0]}; \overrightarrow{[0, 3]}; q; \overrightarrow{[2, 0]} \tag{4.3}$$

where p and q are subtrails of E, neither p nor q contains [0], and either $\mathcal{C}$ or $\mathrm{rev}(\mathcal{C})$ is a consecutive subtrail of one of p or q.

If G has an Euler trail of the form (I), then we say that the embedding G is *type (I) relative to* v and $\mathcal{C}$. If G is not type (I) relative to v and $\mathcal{C}$, and if it has an Euler trail of the form (IIa), we say that G is *type (II) relative to* v and $\mathcal{C}$. Finally, if G is neither type (I) nor type (II) relative to v and $\mathcal{C}$, it must have an Euler trail of the form (IIb), and we say that it is *type (IIb)* *relative to* v *and* $\mathcal{C}$. It is emphasized that the form (without reference to $\mathcal{C}$) of the Euler trail is not enough to specify the type of embedding. For instance, it is possible for G to have an Euler trail as given in equation (4.1), and yet not be of type (I). This would happen if for every possible choice of an Euler trail in this form both $\mathcal{C}$ and its reverse are consecutive subtrails of neither r nor s. We also note that we need only prove lemma 4 for type (I) and type (II) embeddings, relative to a given (but arbitrary) choice of v and $\mathcal{C}$. This follows from the fact that, for G a type (IIb) embedding relative to v and $\mathcal{C}$, reflecting G across the vertical axis at $v = [0]$ results in a type (II) embedding relative to v and $\mathcal{C}'$, where $\mathcal{C}'$ denotes the reflection of $\mathcal{C}$ across the vertical axis.

For either type of embedding, the method of proof is similar. For example, in the case where G has a type (I) Euler trail E, the subtrail r, minus the edges in $r \cap S_{[0]}$, decomposes into a number of consecutive subtrails, $r_1, r_2, \ldots, r_m$, of E. Similarly, $s \setminus (s \cap S_{[0]})$ yields a decomposition, $s_1, s_2, \ldots, s_{m'}$. Since $\mathcal{C}$ (or its reverse) is a consecutive subtrail of E outside $S_{[0]}$, then there is a j such that $\mathcal{C}$ (or its reverse) is a consecutive subtrail of one of r_j or s_j. The goal is to change G within $S_{[0]}$ so that $\deg_{G'}([i]) \leqslant 2, 0 \leqslant i \leqslant 20$, and so that G' has an Euler trail E' containing r_i (or its reverse), for each $i = 1, \ldots, m$, and s_i (or its reverse), for each $i = 1, \ldots, m'$, as consecutive subtrails (not necessarily in the same order as in E). In this case, the trails r_i, $i = 1, \ldots, m$, and s_j, $j = 1, \ldots, m'$, are said to be *essentially preserved* in E' and thus $\mathcal{C}$ (or its reverse) will be a consecutive subtrail of E'. The appropriate altering of $G \cap S_{[0]}$ is accomplished by a detailed case analysis.

4.2.1. G is type (I) relative to v *and* $\mathcal{C}$. By lemma 3, [0] is not a cut vertex. Thus, the removal of any of the four edges incident on [0] does not disconnect G. Let E be a type (I) Euler trail of G relative to v and $\mathcal{C}$.

Consider the set F of four L-shaped subgraphs of $S_{[0]}$ given by $F = \{[4,5] \cup [1,5], [1,6] \cup [2,6], [2,7] \cup [3,7], [3,8] \cup [4,8]\}$, with corresponding corners given by the four vertices [5], [6], [7] and [8]. Letting $\mathcal{L}$ be any L-shape $\mathcal{L} \in F$, we observe that $\mathcal{L} \nsubseteq G$. This is because each of the vertices incident on $\mathcal{L}$ has degree 2 in G, and hence $\mathcal{L} \subseteq G$ implies that the edges of $\mathcal{L}$ are traversed by both r and s, contradicting the fact that E is a trail.

If the edges of L (as in figure 4) form a consecutive subtrail of s (denoted by $L \subset s$), then L is said to produce a *twist* in s if the order of its three vertices is [15], [6] and [16], while traversing s from [2] to [4]; otherwise L produces *no twist* in s.

Using the above facts and definitions, the problem is reduced to the following cases. Solutions are presented in figure 5. Any configurations not shown in figure 5 can be obtained by symmetry arguments from those shown in figure 5.

(1) One of [5], [6], [7] or [8] is unoccupied: a flip transformation yields G' with n edges. Figure 5(a) shows the configuration and solution when [8] is unoccupied.
(2) [5], [6], [7], [8] and an edge from one of the L-shapes in F is occupied: a U-turn transformation yields G' with $n - 2$ edges. Figure 5(b) shows the solution for the case where edge [4, 8] is occupied.
(3) [5], [6], [7] and [8] are occupied, but no edge from any of the L-shapes in F is occupied: hence all four L-shapes BkL, L, Γ and BkΓ must be occupied. Note that when $L \subset r$, the configuration may be reflected through the line $y = x$ in order to obtain the case $L \subset s$, after appropriate relabelling of the vertices. Thus, without loss of generality, suppose $L \subset s$. Therefore, [10, 15] is not occupied, since r and s share no common edge.
   (a) [11, 16] is occupied: figure 5(c) shows the solution G' with $n - 4$ edges.
   (b) [11, 16] is not occupied and
      (i) L produces a twist in s: figure 5(d) yields G' with $n - 2$ edges.

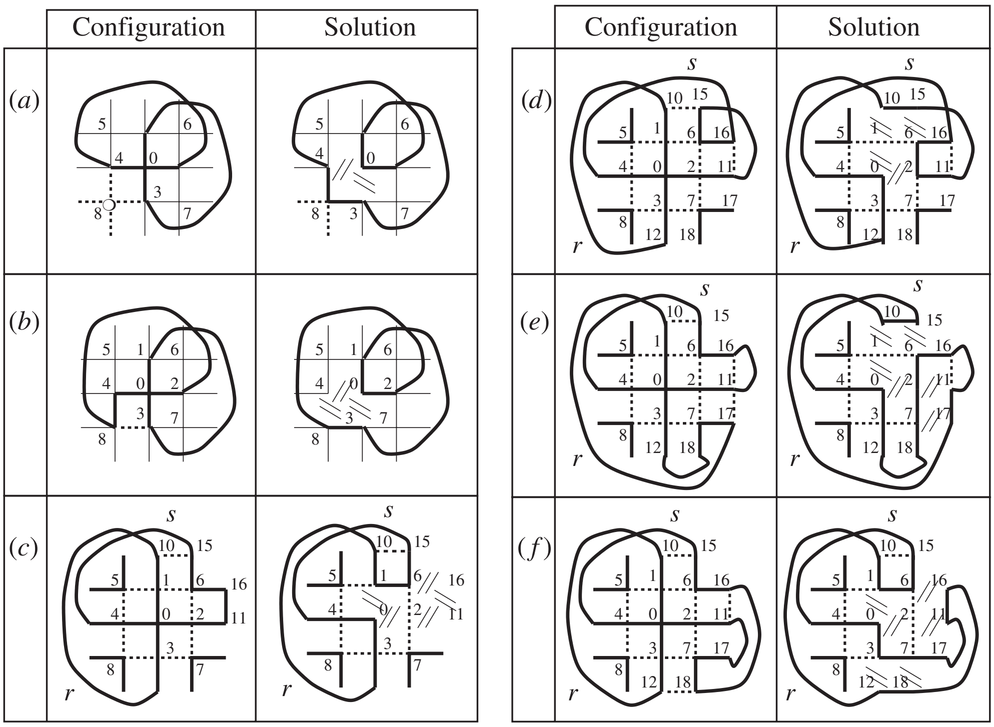

Figure 5. G is type (I) relative to $v = [0]$ and $\mathcal{C}$. (a)–(f): solutions. Bold solid edges are occupied, dashed edges are not occupied, and fine solid edges may or may not be occupied. Double hash marks indicate edges which have been removed, during the solution. In figure (a), the hollow circle indicates an unoccupied vertex.

(ii) L produces no twist in s and $\Gamma \subset r$: then [11, 17] is not occupied, and figure 5(e) yields G' with $n-2$ edges.

(iii) L produces no twist in s and $\Gamma \subset s$: then [12, 18] is not occupied, and figure 5(f) yields G' with $n-2$ edges.

4.2.2. G is type (II) relative to v and $\mathcal{C}$. We note that $E = \mathcal{C}_p \cup \mathcal{C}_q$ is the union of two consecutive closed subtrails containing p and q, respectively, where $\mathcal{C}_p = \overrightarrow{[0, 1]}, p, \overrightarrow{[2, 0]}$ and $\mathcal{C}_q = \overrightarrow{[0, 3]}, q, \overrightarrow{[4, 0]}$. The basic idea is to break $\mathcal{C}_p$ apart from $\mathcal{C}_q$ at their common vertex [0], transforming them into separate closed trails $\mathcal{C}_p \longmapsto t_p$ and $\mathcal{C}_q \longmapsto t_q$. The new trails t_p and t_q are then rejoined to form one single consecutive closed trail. All transformations take place inside $S_{[0]}$, in a manner that reduces the degree of [0] from 4 to 2, introduces no new vertices of degree 4 in the graph, and essentially preserves consecutive subtrails of E outside $S_{[0]}$. The fact that this is accomplished successfully is self-evident in the cases depicted in figures 7 and 10, while more detailed explanations are provided in the cases depicted in figures 9 and 11. For future reference, we define the following edges in table 1, which are labelled in figure 6. The proof involves a case analysis, as outlined below.

(1) At least one of [5] or [7] is unoccupied: a flip transformation yields G' with n edges. Figure 1(a) shows the solution when [5] is unoccupied.

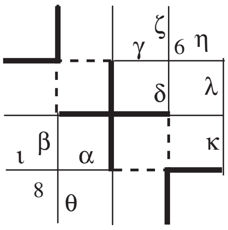

Figure 6. The ten edges α through λ are labelled here.

Table 1. Edges.

α	β	γ	δ	ζ	η	θ	ι	κ	λ
[3, 8]	[4, 8]	[1, 6]	[2, 6]	[6, 15]	[6, 16]	[8, 19]	[8, 20]	[11, 17]	[11, 16]

(2) [5], [7], and one of [4, 5], [1, 5], [2, 7] or [3, 7] are occupied: a U-turn transformation yields G' with $n - 2$ edges. Figure 1(*b*) shows the solution for the case where [4, 5] is occupied.

(3) [5] and [7] are occupied, and none of the edges [4, 5], [1, 5], [2, 7] or [3, 7] are occupied: this implies that BkL and Γ must each be occupied. Now, either BkL and Γ can both be in the same subtrail (either p or q), or else one of BkL and Γ is contained in p, while the other is contained in q.

(a) BkL and Γ are both in the same subtrail: without loss of generality, assume both are contained in p.

(a1) [8] is not occupied: G' with n edges is obtained as shown in figure 7(*a*).

(a2) [8] is occupied:

(a2.1) α and β are occupied: remove the four edges α, β, [0, 3] and [0, 4] to obtain G' with $n - 4$ edges.

(a2.2) One of α or β is occupied: obtain G' with $n - 2$ edges, as shown in figure 7(*b*) for the case where α is occupied.

(a2.3) α and β are unoccupied: therefore, BkΓ is occupied.

(a2.3.1) [6] is unoccupied: obtain G' with n edges, as shown in figure 7(*c*).

(a2.3.2) [6] is occupied: note first that it is not possible to have both γ and δ occupied at once (or else BkL and Γ could not be contained in p).

(a2.3.2.1) One of γ or δ is occupied: obtain G' with $n - 2$ edges as shown in figure 7(*d*) for the case where γ is occupied.

(a2.3.2.2) γ and δ are unoccupied: hence L is occupied.

(a2.3.2.2.1) $L \subset q$: hence both [10, 15] and [11, 16] are unoccupied (otherwise L would be in p). Thus G' with $n - 4$ edges is obtained as shown in figure 7(*e*).

(a2.3.2.2.2) $L \subset p$: note that at most one of [11, 16] or [10, 15] is occupied (otherwise BkL and Γ could not be contained in p).

(a2.3.2.2.2.1) One of [11, 16] or [10, 15] is occupied: G' with $n - 4$ edges is obtained as shown in figure 7(*f*) for the case where [11, 16] is occupied.

(a2.3.2.2.2.2) [11, 16] and [10, 15] are unoccupied: then the three L-shapes, BkL, L and Γ may be met in six possible orders, while traversing p from [1] to [2], as illustrated in cases (I)–(VI) in figure 8. Because of both geometry and the order in which the three L-shapes are traversed, we observe that case (V) may be flipped across the line $y = x$ to obtain case (III), and case (VI) may be

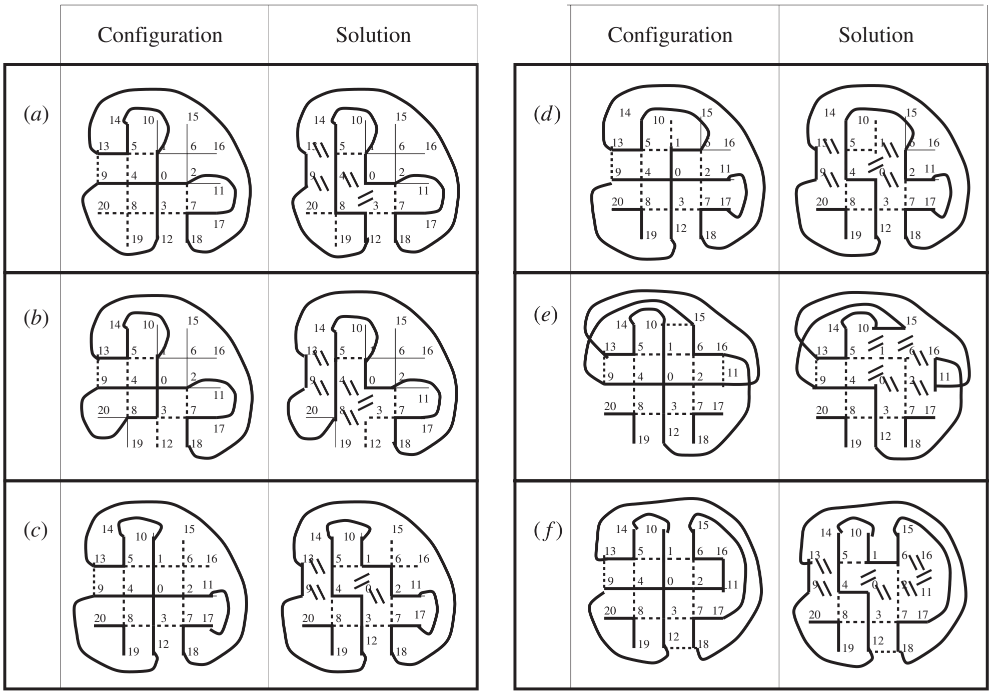

Figure 7. Solutions for cases (3a1)–(3a2.3.2.2.2.1). Bold solid edges are occupied; dashed edges are not occupied; and fine solid edges may or may not be occupied. Double hash marks indicate edges which have been removed, during the solution.

flipped across the line $y = x$ to obtain case (IV). We, therefore, need to solve only the first four cases (I)–(IV). To each of these four cases, as illustrated in figure 9(a), we apply *step (1)*: remove the four edges [0, 1], [0, 2], [2, 11] and [6, 16], and add the two edges [1, 6] and [11, 16]. Step (1) does not affect the closed trail $\mathcal{C}_q$, and hence yields the transformation $t_q = \mathcal{C}_q = \overrightarrow{[0, 3]}, q, \overrightarrow{[4, 0]}$. Step (1) does, however, seal off $\mathcal{C}_p$ into either one or two closed trails, depending on the direction of traversal through L.

(a2.3.2.2.2.2.1) L is traversed from right to left: hence $\mathcal{C}_p$ gets sealed off into one closed trail t_p as in figure 9(b). As illustrated in figure 9(d), we obtain G' with $n - 2$ edges, by connecting t_p to t_q using *step (2)*: remove the two edges [7, 18] and [3, 12], and add the two edges [3, 7] and [12, 18].

(a2.3.2.2.2.2.2) L is traversed from left to right: hence $\mathcal{C}_p$ gets sealed off into two separate closed trails t_{p_1} and t_{p_2} as shown in figure 9(c). In this case each of cases (I)–(IV) in figure 8 must be treated separately, as follows:

(I) Since $\Gamma \subset t_{p_2}$, applying step (2) will connect t_{p_2} and t_q into one closed trail, $t_{q \cup p_2}$; and since BkL $\subset t_{p_1}$, then as illustrated in figure 9(e), we finally obtain G' with $n - 2$ edges, by connecting $t_{q \cup p_2}$ and t_{p_1} using *step (3)*: remove the two edges [5, 13] and [4, 9], and add the two edges [4, 5] and [9, 13].

(II) Since $\Gamma \subset t_{p_1}$, applying step (2) will connect t_{p_1} and t_q into one closed trail, $t_{q \cup p_1}$; and since BkL $\subset t_{p_2}$, then as illustrated in figure 9(f), we obtain G' with $n - 2$ edges, by connecting $t_{q \cup p_1}$ and t_{p_2} using step (3), as described in (I) above.

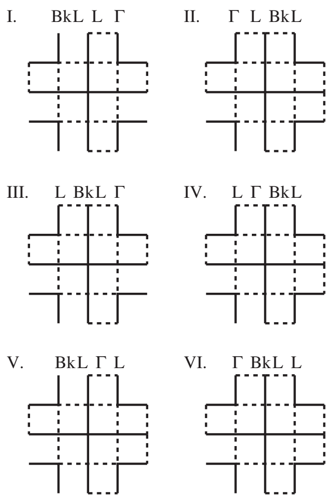

Figure 8. The six subcases for case (3a2.3.2.2.2.2). Solid edges are occupied and dashed edges are not occupied. While traversing p from [1] to [2], the three L-shapes are met in the following order: (I) BkL, L, Γ; (II) Γ, L, BkL; (III) L, BkL, Γ; (IV) L, Γ, BkL; (V) BkL, Γ, L; and (VI) Γ, BkL, L.

(III) Since $\Gamma \subset t_{p_2}$, applying step (2) will connect t_{p_2} and t_q into one closed trail, $t_{q\cup p_2}$; and since BkL $\subset t_{p_2}$ also, then as illustrated in figure 9(g), we obtain G' with $n-2$ edges, by connecting $t_{q\cup p_2}$ and t_{p_1} as follows: remove the two edges [5, 14] and [1, 10], and add the two edges [1, 5] and [10, 14].

(IV) Since $\Gamma \subset t_{p_2}$ and BkL $\subset t_{p_2}$, the same steps apply, as described in (III) above.

(b) BkL and Γ both come from different subtrails: without loss of generality, assume that BkL is contained in p, while Γ is in q.

(b1) One of [6] or [8] is unoccupied: obtain G' with n edges, as shown in figure 10(a) for the case where [6] is unoccupied.

(b2) [6] and [8] are occupied. This case is split into two major subcases, depending on the occupational states of the four edges α, β, γ or δ, as follows:

(b2.I) Edges α, β, γ and δ are unoccupied: hence L and BkΓ are both occupied.

(b2.I.1) $L \subset q$: hence [10, 15] and [11, 16] are unoccupied. Obtain G' with $n-4$ edges as in case (3a2.3.2.2.1), following figure 7(e).

(b2.I.2) $L \subset p$: there are two subcases depending on the order in which we meet BkL and L, while traversing p from [1] to [2]:

(b2.I.2.i) First BkL, then L: as illustrated in figure 11(a), apply *step (1)*: remove the four edges [0, 1], [0, 2], [1, 10] and [6, 15], and add the two edges [10, 15] and [2, 6]. Step (1) does not affect the closed trail $\mathcal{C}_q$ and hence yields the transformation $t_q = \mathcal{C}_q = \overrightarrow{[0, 3]}, q, \overrightarrow{[4, 0]}$. Step (1) does, however, seal off $\mathcal{C}_p$ into either one or two closed trails, depending on the direction of traversal through L.

(b2.I.2.i.1) L is traversed from right to left: $\mathcal{C}_p$ gets sealed off into one closed trail t_p, as in figure 11(b). As illustrated in figure 11(d), we obtain G' with $n-2$

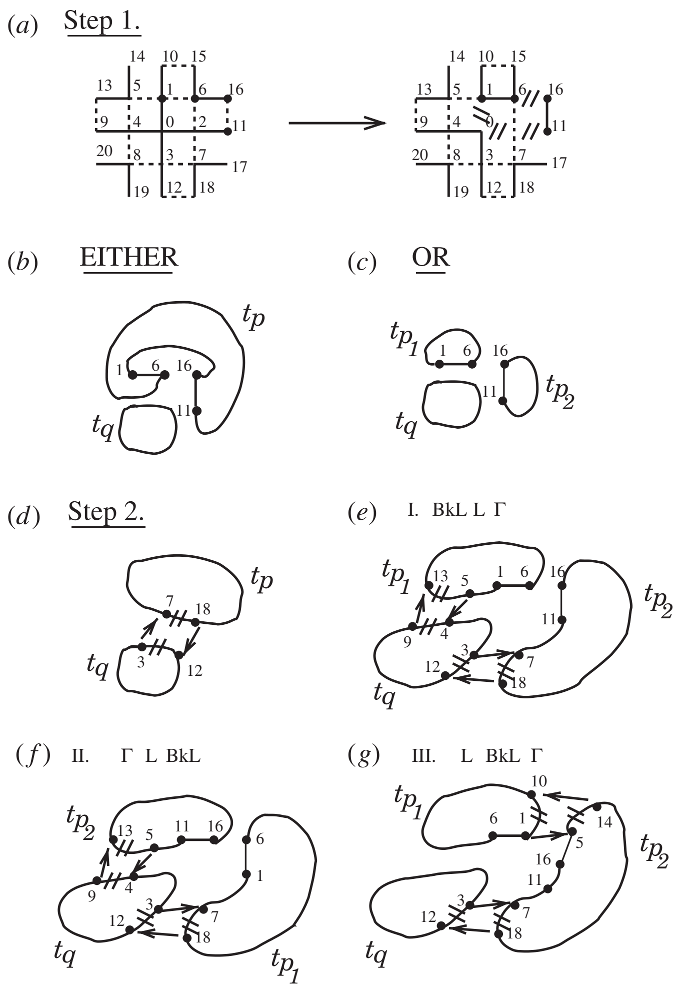

Figure 9. Solutions for case (3a2.3.2.2.2.2). Solid edges are occupied and dashed edges are not occupied. Double hash marks indicate edges which have been removed during the solution.

edges, by connecting t_p to t_q using *step 2*: remove the two edges [7, 17] and [2, 11], and add the two edges [2, 7] and [11, 17].

(b2.I.2.i.2) L is traversed from left to right: $\mathcal{C}_p$ gets sealed off into two separate trails t_{p_1} and t_{p_2} as shown in figure 11(*c*). Since $[2, 11] \subset t_{p_2}$, applying step (2) from (b2.I.2.i.1) above, will connect t_{p_2} and t_q into one closed trail, $t_{q\cup p_2}$; and since $\text{BkL} \subset t_{p_1}$ then, as illustrated in figure 11(*e*), we finally obtain G' with $n-2$ edges, by connecting $t_{q\cup p_2}$ and t_{p_1} as follows: remove the two edges [5, 13] and [4, 9], and add the two edges [4, 5] and [9, 13].

(b2.I.2.ii) First L, then BkL: then, as illustrated in figure 11(*f*), we apply *step (a)*: remove the four edges [0, 1], [0, 2], [2, 11] and [6, 16], and add the two edges [11, 16] and [1, 6]. Step (a) does not affect the closed trail $\mathcal{C}_q$ and hence yields the transformation $t_q = \mathcal{C}_q = [\overrightarrow{0, 3}], q, [\overrightarrow{4, 0}]$. Step (a) does, however, seal off $\mathcal{C}_p$ into either one or two closed trails, depending on the direction of traversal through L.

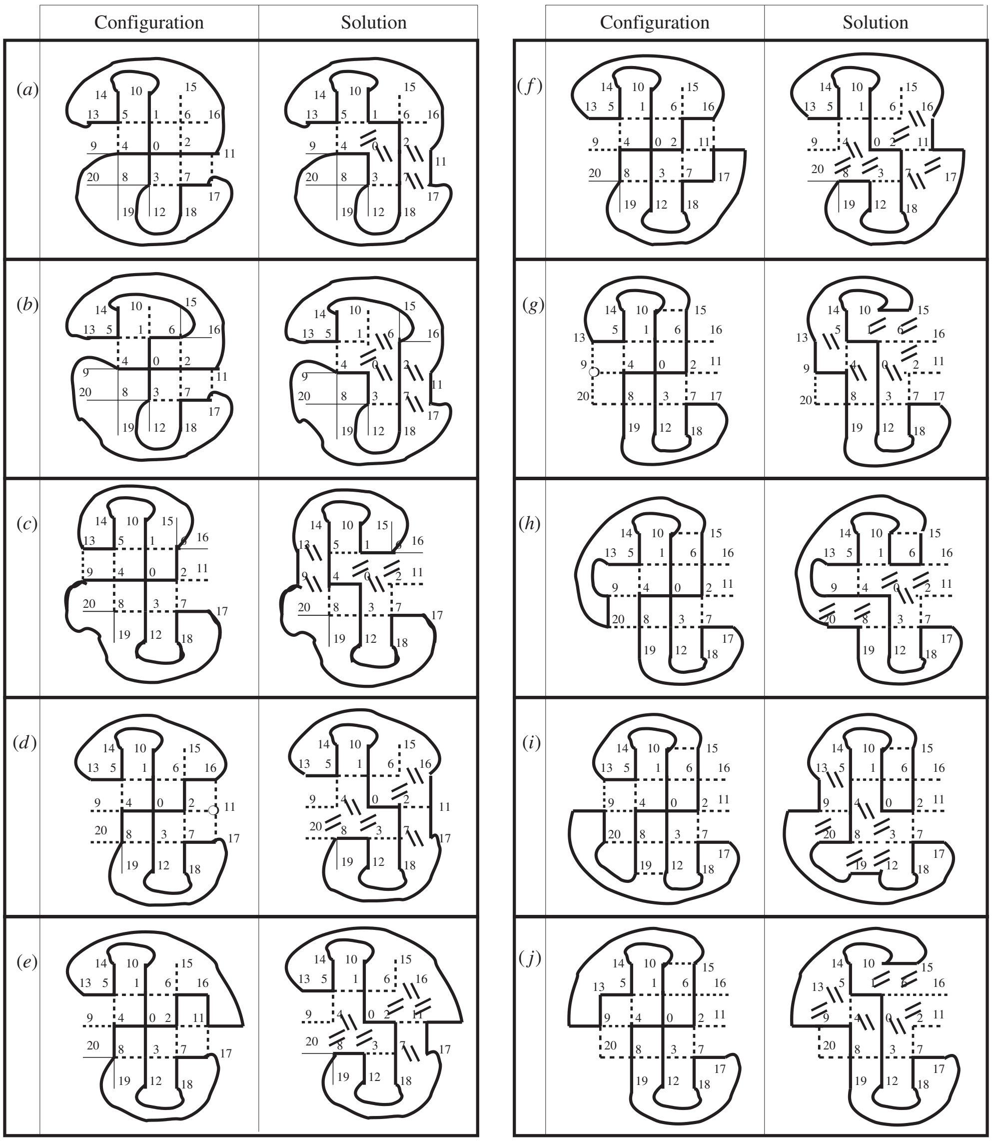

Figure 10. Solutions for case (3b). Bold solid edges are occupied, dashed edges are not occupied, and fine solid edges may or may not be occupied. Double hash marks indicate edges which have been removed during the solution. In figures (*d*) and (*g*), a hollow circle indicates an unoccupied vertex.

(b2.I.2.ii.1) L is traversed from right to left: $\mathcal{C}_p$ gets sealed off into one closed trail t_p as in figure 11(*g*). As illustrated in figure 11(*i*), we obtain G' with $n-2$ edges, by connecting t_p to t_q using *step (b)*: remove the two edges [4, 9] and [5, 13], and add the two edges [4, 5] and [9, 13].

(b2.I.2.ii.2) L is traversed from left to right: $\mathcal{C}_p$ gets sealed off into two separate closed trails t_{p_1} and t_{p_2} as shown in figure 11(*h*). Since BkL $\subset t_{p_2}$, applying step (b) from (b2.I.2.ii.1) above, will connect t_{p_2} and t_q into one closed trail, $t_{q\cup p_2}$; and since $[1, 10] \subset t_{p_1}$ then, as illustrated in figure 11(*j*), we finally obtain

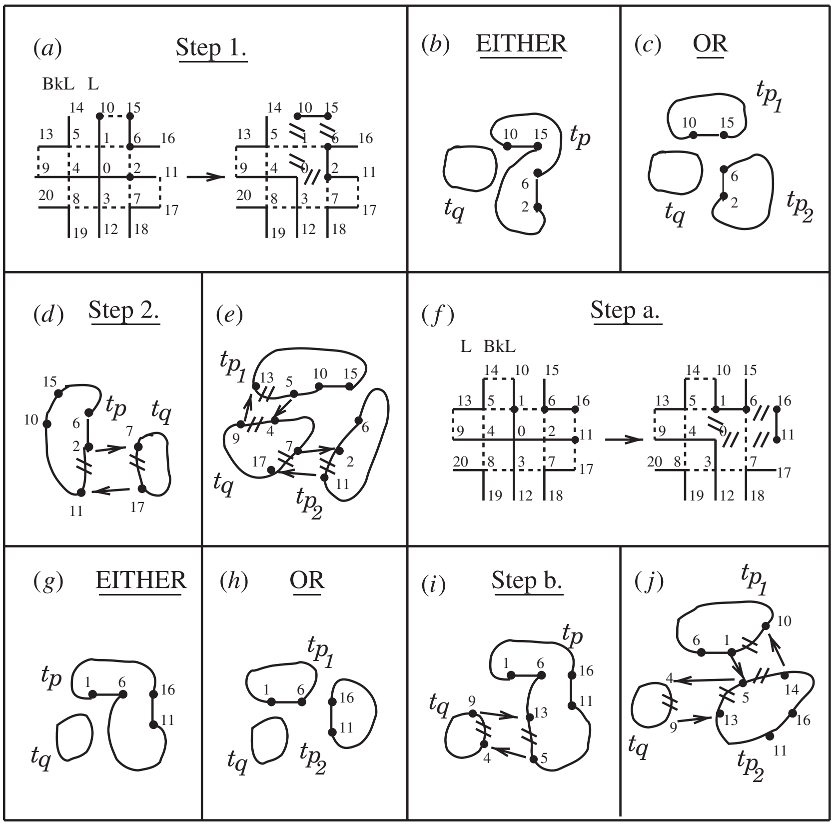

Figure 11. Solutions for case (3b.2). Solid edges are occupied and dashed edges are not occupied. Double hash marks indicate edges which have been removed during the solution.

G' with $n-2$ edges, by connecting $t_{q\cup p_2}$ and t_{p_1} as follows: remove the two edges [1, 10] and [5, 14], and add the two edges [1, 5] and [10, 14].

(b2.II) At least one of the edges α, β, γ and δ is occupied: as a shorthand notation, we mention that an edge (or vertex) is occupied by setting it equal to 1; likewise, an edge (or vertex) which is not occupied is set equal to 0.

The four possible subcases are as follows:

(b2.II.1) $\gamma = 1$: then $\delta = 0$ (or else BkL could not be in p). Obtain G' with $n-2$ edges as shown in figure 10(b).

(b2.II.2) $\alpha = 1$: then $\beta = 0$ (or else Γ could not be in q). Rotate the figure by 180°, and after appropriate relabelling of the vertices, follow the same steps as in (b2.II.1) above.

(b2.II.3) $\alpha = \gamma = 0$ with $\beta = 0$: then $\delta = 1$, so we obtain G' with $n-2$ edges as shown in figure 10(c).

(b2.II.4) $\alpha = \gamma = 0$ with $\beta = 1$: assume, without loss of generality, that $\delta = 1$ (for if not, then rotation of the graph by 180° and appropriate relabelling would yield case (b2.II.3), above). Next, observe that the case where $\iota = 1$ and $[9] = x \in \{0, 1\}$ it may be rotated by 180° to yield the case where $\eta = 1$ and $[11] = x$, after appropriate relabelling. Therefore, this case further splits into three subcases, as follows:

(b2.II.4.1) $\eta = 1$ and $[11] = 0$: obtain G' with $n-2$ edges as in figure 10(d).

(b2.II.4.2) $\eta = 1$ and $[11] = 1$: then either $\lambda = 1$ or $\kappa = 1$, but not both, else p and q would share a common edge.

(b2.II.4.2.1) $\lambda = 1$: obtain G' with $n-4$ edges as in figure 10(e).

(b2.II.4.2.2) $\kappa = 1$: obtain G' with $n-4$ edges as in figure 10(f).

(b2.II.4.3) $\eta = 0$ and $\iota = 0$:

(b2.II.4.3.1) $[9] = 0$: obtain G' with $n-2$ edges as in figure 10(g).

(b2.II.4.3.2) $[9] = 1$:

(b2.II.4.3.2.1) $[9, 20]$ is occupied and $[9, 20] \subset p$: obtain G' with $n-2$ edges as in figure 10(h).

(b2.II.4.3.2.2) $[9, 20]$ is occupied and $[9, 20] \subset q$: obtain G' with $n-2$ edges as in figure 10(i).

(b2.II.4.3.2.3) $[9, 20]$ is unoccupied: therefore, $[9, 13]$ is occupied, and we obtain G' with $n-4$ edges as in figure 10(j). □

5. Removal of the top vertex of degree 4 in a closed Eulerian graph

Given a plane embedding G, let $\mathbb{V} = \{v \in V_G : \deg_G(v) = 4\}$. We define the *top vertex of degree 4 in G*, to be the topmost vertex of the set of rightmost vertices contained in $\mathbb{V}$. For this discussion, we assume that G is an n-tau, for $\tau \in \mathcal{G}_4^0(k)$, for some $k \geqslant 1$, and let $[A]$ be the top vertex of degree 4 in G.

The term *box* will be used to refer to a subgraph of $\mathbb{Z}^2$ induced by a vertex set of the form $R = \{(x, y) \in \mathbb{Z}^2 | a_1 \leqslant x \leqslant b_1, a_2 \leqslant y \leqslant b_2\}$. A *unit box* is one for which $b_1 - a_1 = b_2 - a_2 = 1$. Since R completely determines the box, we will typically use the vertex set R to refer to the box. Given a vertex $v \in \mathbb{Z}^2$, the notation $v + R$ will be used to refer to the box whose vertex set is obtained by translating each of the vertices in R by the vector v. The inside and outside of a box R are defined in a manner consistent with the definition given earlier for the inside and outside of $S_{[0]}$.

Label the four corners of a unit box B in $\mathbb{Z}^2$ by $[Q_i]$, for $0 \leqslant i \leqslant 3$, clockwise starting with $[Q_0]$ in the lower left-hand corner. For each i, let $j_i \equiv (i+1) \bmod 4$, and define the edge E_i between the two adjacent vertices $[Q_i]$ and $\left[Q_{j_i}\right]$ as $E_i = \left[Q_i, Q_{j_i}\right]$ for $0 \leqslant i \leqslant 3$. Then B is the union of these edges, $B = \cup_{i=0}^3 E_i$, see figure 12(a). For the purposes of this discussion we assume, for $0 \leqslant i \leqslant 3$, that $[Q_i]$ is in the *safe-zone*, the half-plane of vertices strictly greater (lexicographically) than $[A]$ (the shaded region shown in figure 12(a)). From this, it follows that $[A] \notin \cup_{i=0}^3\{[Q_i]\}$, and that $\deg_G([Q_i]) \in \{0, 2\}$, for $0 \leqslant i \leqslant 3$.

Lemma 5. *B being in the safe-zone implies that $B \not\subset G$, since otherwise $[A]$ would be disconnected from B in G (see figure 12(b)).*

Define the *U-turn* U_i by the three-edged U-shape $U_i = B \backslash E_i$. If $U_i \subset G$, then by lemma 5, $E_i \notin G$. Hence the U-turn reduction on $U_i \subset G$, which replaces the three edges of U_i by the single edge E_i is well defined and yields the $(n-2)$-tau $G_{u_i} = (G \backslash U_i) \cup E_i$.

Next, define L_i to be the L-shape $L_i = E_i \cup E_{j_i} \subset B$, where E_i and E_{j_i} are the two edges incident on $\left[Q_{j_i}\right]$. If $L_i \subset G$ and $(B \backslash L_i) \cap \mathcal{E}_G = \phi$ then the flip of L_i with respect to B, which replaces the two edges of L_i by the two edges $B \backslash L_i$ is well defined and yields the n-tau $G_{L_i} = (G \backslash L_i) \cup (B \backslash L_i)$.

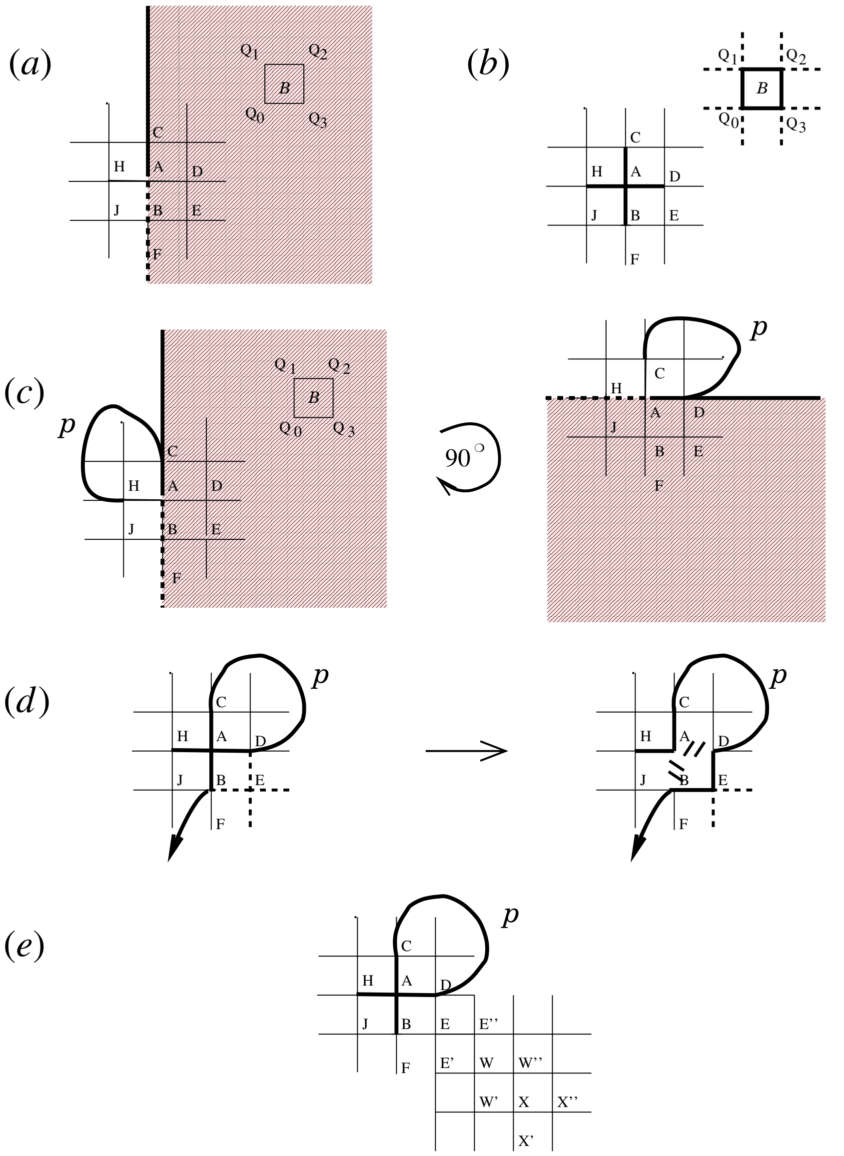

Figure 12. $[A]$ is the top vertex of degree 4 of G. (a) The unit box B in the safe-zone, indicated by the shaded region. (b) $B \not\subset G$. (b)–(e) Bold solid (dashed) lines indicate occupied (unoccupied) edges. Light solid lines indicate edges which may or may not be occupied. (c) If G is a type (I) or (IIb) embedding, obtain $G_1 = \pi G$ by a 90° clockwise rotation of G, followed by appropriate relabelling. (d) Solution for case (1) $\deg([E]) = 0$, base step. Double hash marks indicate edges which have been removed during solution. (e) Label eight additional vertices to facilitate the discussion of case (2) $\deg([E]) = 2$.

Lemma 6. *Let $\tau \in \mathcal{G}_4^0(k)$, for some $k \geqslant 1$, and G be an n-tau. Let R be the box $R = \{(x, y) \in \mathbb{Z}^2 \mid -5 \leqslant x \leqslant 5, -5 \leqslant y \leqslant 5\}$, and suppose that $[A]$ is the top vertex of degree 4 in G. Then it is possible, by only altering edges and vertices inside the box $[A] + R$,* *centred on $[A]$, to construct a new closed Eulerian graph G' with m edges and k' vertices of degree 4, such that all the following conditions are met:*

(i) $\deg_{G'}([A]) = 2$.
(ii) $G' = G$ *outside the box* $[A] + R$.
(iii) $m \in \{n-4, n-2, n\}$ *and* $0 \leqslant k' < k$.
(iv) No new vertices of degree 4 are introduced: $\{w \in G' \mid \deg_{G'}(w) = 4\} \subset \{w \in G \mid \deg_G(w) = 4\}$. *Hence, for* $k' \geqslant 1$, $v \in \{w \in G' \mid \deg_{G'}(w) = 4\} \Rightarrow v < [A]$ *(lexicographically).*
(v) Given an open trail $\mathcal{C}$ *of* G *consisting of edges outside* $[A] + R$ *and such that* G *contains an Euler trail which contains* $\mathcal{C}$ *as a consecutive subtrail, then* G' *also has an Euler trail which contains* $\mathcal{C}$ *as a consecutive subtrail.*

Proof. The basic idea of the proof is to reduce the degree of $[A]$ by performing a sequence of flip transformations and/or U-turn transformations until there is one less vertex of degree 4 in the embedding or the top vertex of degree 4 in the embedding has been moved so that it becomes an isolated vertex of degree 4. In the latter case, lemma 4 can be applied to remove the resulting isolated vertex of degree 4. Initially, the embedding may need to be rotated and relabelled so that the transformations can be applied in the south-east direction.

The first step of the proof depends on whether G is a type (I), (II), or (IIb) embedding relative to $[A]$ and $\mathcal{C}$, with the types as defined in section 4.2. In particular, if G is a type (I) or (IIb) embedding we first rotate G clockwise by 90°, and then relabel the embedding as shown in figure 12(c) to obtain G_1. Note that the safe-zone is rotated as well. If G is type (II), set $G_1 = G$. In either case, let π be the operator that transforms G to G_1, i.e. $G_1 = \pi G$.

From equations (4.1)–(4.3), G_1 has an Euler trail E_1 such that $\hat{L} = ([B, A], [A, D])$ is a consecutive subtrail of E_1. Thus by lemma 2, the edges in $\hat{L}$ can be removed from G_1 without disconnecting it.

We next note a useful result for the case that $G_1 = G$ is type (II) relative to $[A]$ and $\mathcal{C}$. For this type of embedding, there is an Euler trail $\hat{E}$ of G_1 in the following form,

$$\hat{E} = [\overrightarrow{A, C}]; \quad p; \quad [\overrightarrow{D, A}]; \quad [\overrightarrow{A, B}]; \quad q; \quad [\overrightarrow{H, A}] \tag{5.1}$$

with $\mathcal{C}$ or $\mathrm{rev}(\mathcal{C})$ a consecutive subtrail of one of p or q. The useful result is that p and q cannot share any common vertices in $[A] + R$. To see this, suppose that there exists $[I] \in [A]+R$ such that $[I]$ is used in both p and q. Then another Euler trail of G_1 is as follows: $E_{\text{new}} = [\overrightarrow{A, C}]$; p_1; $\mathrm{rev}(q_1)$; $[\overrightarrow{B, A}]$; $[\overrightarrow{A, D}]$; $\mathrm{rev}(p_2)$; q_2; $[\overrightarrow{H, A}]$, where $p = (p_1; [I]; p_2)$ and $q = (q_1; [I]; q_2)$. Since $[I] \in [A] + R$, it follows that $\mathcal{C}$ (or its reverse) must be a consecutive subtrail of one of the four subtrails, p_1, p_2, q_1 or q_2. But, this means that G is type (I) relative to $[A]$ and $\mathcal{C}$, which is a contradiction.

One important consequence of this is that the edges $e_1 = [B, E]$ and $e_2 = [D, E]$ cannot both be occupied if $G_1 = G$. Since otherwise, p and q must share at least one of the three vertices $[D]$, $[E]$ or $[B]$, for any way the edges e_1 and e_2 are traversed by the two subtrails p and q.

In fact, the edges $[B, E]$ and $[D, E]$ cannot both be occupied in G_1 for the case $G_1 \neq G$, either. If we assume to the contrary that $e_1 \cup e_2 \subset G$, then $\deg_{G_1}[D] = \deg_{G_1}[E] = \deg_{G_1}[B] = 2$, since all three of these vertices are in the safe-zone. This would imply that removal of the edges in $\hat{L}$ disconnects G_1 which is a contradiction.

We are now ready to proceed with the case analysis of the proof.

(1) $\deg_{G_1}([E]) = 0$: perform the *base-step*: remove the edges in $\hat{L}$ and add the L-shape $[B, E] \cup [D, E]$, see figure 12(d), to obtain $\pi G'$ with n edges.

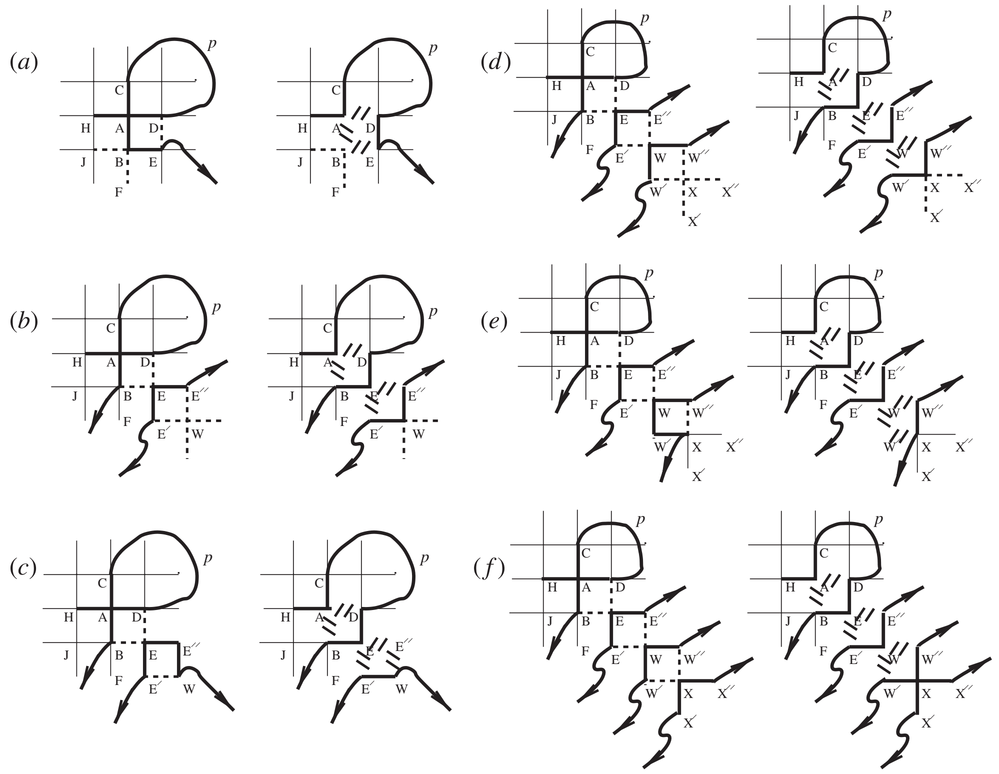

Figure 13. Solution for some of the cases arising in section 5.1, where $\deg_{G_1}([B]) = 2$. (*a*) Case (A), for $[B, E]$ occupied. (*b*) Case (B.1), $\deg_{G_1}([W]) = 0$. (*c*) Case (B.2, 1), for $[W, E'']$ occupied. (*d*) Case (B.2, 2.1), $\deg_{G_1}([X]) = 0$. (*e*) Case (B.2, 2.2a), for $[W', X]$ occupied. (*f*) Case (B.2, 2.2b) results in the isolated vertex of degree 4, X.

(2) $\deg_{G_1}([E]) = 2$: to facilitate discussion of this case, eight additional vertices are labelled, as shown in figure 12(*e*). Note that each of these eight additional vertices either have degree 0 or 2, since they are each in the safe-zone.

The solution splits into two subcases: either $\deg_{G_1}([B]) = 2$, or $\deg_{G_1}([B]) = 4$. These subcases are outlined in sections 5.1 and 5.2.

5.1. $\deg_{G_1}([B]) = 2 = \deg_{G_1}([E])$

This splits into two further subcases (A) and (B), as follows:

(A) One of $[D, E]$ or $[B, E]$ is occupied: perform a U-turn transformation on this edge to obtain $\pi G'$ with $n - 2$ edges and $k - 1$ vertices of degree 4. This step is illustrated in figure 13(*a*), for $[B, E]$ occupied.

(B) $[B, E]$ and $[D, E]$ are unoccupied: in this case, $\Gamma = [E, E'] \cup [E, E''] \subset G_1$.

(B.1) $\deg_{G_1}([W]) = 0$: obtain $\pi G'$ with n edges from G_1, as in figure 13(*b*), by performing the following steps: a flip on Γ with respect to the unit box B_1 (with corners given by $[E]$, $[E']$, $[E'']$ and $[W]$) and then the base-step.

(B.2) $\deg_{G_1}([W]) = 2$: by lemma 5, it is impossible to have both edges $[W, E']$ and $[W, E'']$ occupied simultaneously. Therefore, this case further splits into the following subcases:

(1) One edge, $e \in \{[W, E'], [W, E'']\}$ is occupied: perform a U-turn reduction on $U = \Gamma \cup \{e\}$, with respect to the unit box B_1 followed by the base-step, to obtain $\pi G'$ with $n - 2$ edges. This step is illustrated in figure 13(c), when $[W, E'']$ is occupied.

(2) $[W, E']$ and $[W, E'']$ are unoccupied: then $\tilde{\Gamma} = [W, W'] \cup [W, W''] \subset G_1$.

(2.1) $\deg_{G_1}([X]) = 0$: as illustrated in figure 13(d), $\pi G'$ with n edges is obtained by following the steps: a flip on $\tilde{\Gamma}$ with respect to the unit box B_2 (with corners given by $[W]$, $[W']$, $[W'']$ and $[X]$), a flip on Γ with respect to B_1, and then the base-step.

(2.2) $\deg_{G_1}([X]) = 2$: by lemma 5, it is not possible to have both $[W', X]$ and $[W'', X]$ occupied simultaneously.

(a) One edge, $f \in \{[W', X], [W'', X]\}$ is occupied: perform a U-turn reduction on $\tilde{U} = \tilde{\Gamma} \cup \{f\}$, with respect to the unit box B_2, then perform a flip on Γ with respect to B_1, and finally perform the base-step. The resulting $\pi G'$ has $n - 2$ edges. This step is illustrated in figure 13(e), when $[W', X]$ is occupied.

(b) $[W', X]$ and $[W'', X]$ are unoccupied: in this case, we know that $\tilde{\tilde{\Gamma}} = [X, X'] \cup [X, X''] \subset G_1$. So, first perform a flip on $\tilde{\Gamma}$ with respect to B_2, second perform a flip on Γ with respect to B_1, and lastly perform the base-step to obtain G_2, an embedding of a closed k-graph with n edges. The vertex $[A]$ of degree 4, has been moved from $[A]$ in G to $[X]$ in G_2, and $[X]$ is now an isolated vertex of degree 4 in G_2 (see figure 13(f)). Thus, lemma 4 can be applied at $[X]$ to obtain $\pi G'$ with m edges, where $m \in \{n, n-2, n-4\}$.

5.2. $\deg_{G_1}([B]) = 4$ *(hence, necessarily,* $\deg_G([E]) = 2$ *and* $G_1 = G$ *is type (II) relative to* $[A]$ *and* $\mathcal{C}$*).*

Note that this means $k \geqslant 2$ and $[D, E]$ is not in G_1. Since G_1 is a type (II) embedding, there is an Euler trail of G_1, $\hat{E}$, in the form given by equation (5.1). Observe that neither $[H]$ nor $[B]$ can appear in p, since otherwise p and q would share a common vertex. It follows from $[B, E] \subset G_1$ that $[E]$ is in q. Also, since $([D, A]; [A, B])$ is a consecutive subtrail of $\hat{E}$, these edges can be removed from G_1 and the edge $[D, E]$ can be added to obtain G_2 which has an open Euler trail in the form

$$\hat{E}_2 = [B]; q_1; [E]; q_2; [\overrightarrow{H, A}]; [\overrightarrow{A, C}]; p; [\overrightarrow{D, E}] \tag{5.2}$$

where $q = (q_1; [E]; q_2)$ and $\mathcal{C}$ (or its reverse) is a consecutive subtrail of one of p, q_1 or q_2. This leads to two subcases, outlined below:

(A) $[\overrightarrow{E, B}]$ is the first edge of q_2: in this case, obtain G', with $n - 2$ edges and $k - 2$ vertices of degree 4, by removing the edge $[E, B]$ from G_2; note that then G' has a closed Euler trail in the form $([\overrightarrow{A, C}]; p; [\overrightarrow{D, E}]; \mathrm{rev}(q_1); [B]; q_2'; [\overrightarrow{H, A}])$, where $q_2' = q_2 - [\overrightarrow{E, B}]$. Basically G' is obtained from G_1 by performing a U-turn transformation, as in figure 14(a).

(B) $[\overrightarrow{B, E}]$ is the last edge of q_1: remove $[B, E]$ from G_2 to obtain the graph G_3. If $q_1 = [\overrightarrow{B, E}]$, then set $G' = G_3$. Otherwise, note that G_3 consists of two edge-disjoint closed subtrails of G_2, $\hat{E}_J$ and $\hat{E}_H$, which together use all the edges of G_3 (i.e. all but three edges of G_1) exactly once, where

$$\hat{E}_J = [J]; q_1'; [\overrightarrow{B, J}] \tag{5.3}$$

with q_1' being $q_1 - [B, E] - [B, J]$ except that it is traversed in the direction $[J]$ to $[B]$, and

$$\hat{E}_H = [\overrightarrow{H, A}]; [\overrightarrow{A, C}]; p; [\overrightarrow{D, E}]; q_2; [H]. \tag{5.4}$$

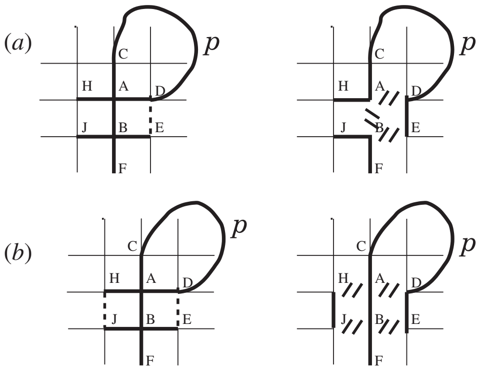

Figure 14. Figures for cases in section 5.2, $\deg_{G_1}([B]) = 4$: (*a*) cases (A) and (Bii); (*b*) case (Bi).

We consider two subcases depending on whether or not the edge $[J, H]$ is occupied in G_1.

(i) $[J, H]$ unoccupied in G_1: obtain G' with $n - 2$ edges and $k - 2$ vertices of degree 4 from G_1 as in figure 14(*b*). This is equivalent to $G' = G_3 \cup \{[A, B], [J, H]\} - \{[H, A], [J, B]\}$ and hence G' has a closed Euler trail of the form $(\overrightarrow{[A, C]}; p; \overrightarrow{[D, E]}; q_2; \overrightarrow{[H, J]}; q_1'; \overrightarrow{[B, A]})$.

(ii) $[J, H]$ occupied in G_1: G_3 is thus connected since $\hat{E}_J$ and $\hat{E}_H$ both intersect at the vertex $[I] \equiv [H]$ or $[J]$, depending, respectively, on whether $[J, H]$ is used in $\hat{E}_J$ or in $\hat{E}_H$. Also, since $\mathcal{C}$ does not intersect $[A] + R$, a closed Euler trail of G_3 which contains $\mathcal{C}$ (or its reverse) as a consecutive subtrail can be obtained as follows: follow $\hat{E}_H$ until q_2 meets $[I]$, then insert a cyclic permutation of $\hat{E}_J$ going from $[I]$ back to $[I]$, and finally continue on q_2 from $[I]$ to the end of q_2. Hence $G' = G_3$ with $n - 2$ edges and $k - 2$ vertices of degree 4 and is essentially obtained from G_1 as in figure 14(*a*). □

6. Removing the top vertex in an open Eulerian graph

Corollary 6. *Let $\tau \in \mathcal{G}_4^2(K)$, for some $K \geqslant 1$, and $\mathbb{G}$ be an N-tau. Let R and $\tilde{R}$ be the boxes $R = \{(x, y) \in \mathbb{Z}^2 | -5 \leqslant x \leqslant 5, -5 \leqslant y \leqslant 5\}$ and $\tilde{R} = \{(x, y) \in \mathbb{Z}^2 | -6 \leqslant x \leqslant 6, -6 \leqslant y \leqslant 6\}$, and suppose that $[A]$ is the top vertex of degree 4 in $\mathbb{G}$. Then it is possible, by only altering edges and vertices inside the box $[A] + \tilde{R}$, centred on $[A]$, to construct a new open Eulerian graph $\mathbb{G}'$ with m edges and k' vertices of degree 4, such that all the following conditions are met:*

(i) $\deg_{\mathbb{G}'}([A]) = 2$.
(ii) $\mathbb{G}' = \mathbb{G}$, *outside the larger box* $[A] + \tilde{R}$.
(iii) $\mathbb{G}' = \mathbb{G}$, *outside the smaller box* $[A] + R$, *for all but at most two edges, which may have been deleted from* $\mathbb{G}$ *to obtain* $\mathbb{G}'$.
(iv) $\max(4, N - 220) \leqslant m \leqslant N$ *and* $0 \leqslant k' < K$.
(v) No new vertices of degree 4 are introduced: $\{w \in \mathbb{G}' \mid \deg_{\mathbb{G}'}(w) = 4\} \subset \{w \in \mathbb{G} \mid \deg_{\mathbb{G}}(w) = 4\}$. *Hence, for* $k' \geqslant 1$, $v \in \{w \in \mathbb{G}' \mid \deg_{\mathbb{G}'}(w) = 4\} \Rightarrow v < [A]$, *in the lexicographical ordering of vertices in* $\mathbb{Z}^2$.

Proof. Label the two odd vertices of $\mathbb{G}$ by a and b, so that $[a] < [b]$, in lexicographical order. For the given N-tau $\mathbb{G}$, prescribe an Euler trail, $E_{\mathbb{G}}$ from [a] to [b]. Then $E_{\mathbb{G}}$ will

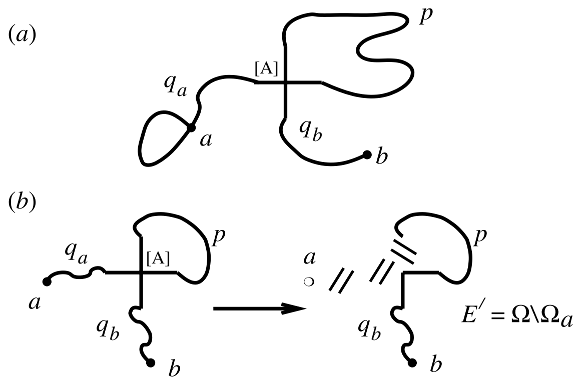

Figure 15. (a) $E_{\mathbb{G}} = [a]; q_a; [A]; p; [A]; q_b; [b]$, where $\deg([a]) \in \{1, 3\}$, $\deg([b]) \in \{1, 3\}$, and the subtrails q_a and q_b may or may not intersect with p or with each other. (b) Case (A), when Ω_a lies entirely inside $[A] + R$, results in the trail $E' = \Omega\backslash\Omega_a$.

necessarily be of the form $E_{\mathbb{G}} = [a]; q_a; [A]; p; [A]; q_b; [b]$, where $\deg_{\mathbb{G}}([a]) \in \{1, 3\}$ and $\deg_{\mathbb{G}}([b]) \in \{1, 3\}$, and the subtrails q_a and q_b may or may not intersect with the subtrail p, or with each other (see figure 15(a)). The approach will be to obtain an open trail E' from $E_{\mathbb{G}}$, whose underlying graph will provide us with the desired $\mathbb{G}'$, satisfying the conditions of corollary 6. In symbols, we let $\mathbb{G}' = \text{u.g.}(E')$ denote the underlying graph of E' obtained by ignoring the orientation of the edges of E'.

First, set $\Omega := E_{\mathbb{G}}$, and for each $v \in \{a, b\}$, denote by Ω_v the consecutive subtrail of Ω with $q_v \subset \Omega_v$ such that Ω_v contains exactly one more edge than q_v. In particular, $\deg_{\Omega_v}([A]) = 2$. Next, one of the following two mutually exclusive cases will occur, from which we obtain E'.

(A) One of Ω_a or Ω_b lies entirely inside the box $[A]+R$: in this case, if Ω_a lies entirely inside $[A]+R$, then set $E' := \Omega\backslash\Omega_a$ and we are done; otherwise Ω_b lies entirely inside $[A]+R$, so set $E' := \Omega\backslash\Omega_b$, and we are done. See figure 15(b).

(B) Ω_a and Ω_b each have edges outside the box $[A]+R$: in this case, the first goal is to ensure that both the odd vertices $[a]$ and $[b]$ are outside the box $[A]+R$. This is done as follows. If $[a]$ lies inside $[A]+R$, then modify Ω by removing consecutive first edges from it, until the first vertex of Ω lies outside $[A]+R$, and then move the old label a to this new first vertex. Next, if $[b]$ lies inside $[A]+R$, modify Ω by removing consecutive last edges from it, until the last vertex of Ω lies outside $[A]+R$, and then move the old label b to this new last vertex. In this way, we have obtained from $E_{\mathbb{G}}$ an n-edge open trail Ω in $\mathbb{Z}^2$ with k vertices of degree 4, where $\max(16, N-206) \leqslant n \leqslant N$, $1 \leqslant k \leqslant K$, $[A]$ is the top vertex of degree 4 in Ω, $\Omega = E_{\mathbb{G}}$ outside $[A]+\tilde{R}$, and the two odd vertices $[a]$ and $[b]$ lie outside $[A]+R$. Thus, there exists an open trail $\mathcal{C}$ in $\mathbb{R}^2$ but outside $[A]+R$, with its two odd vertices given by $[a]$ and $[b]$, such that $\Omega\cup\mathcal{C}$ is a closed Euler trail in $\mathbb{R}^2$ of the planar graph $G = \text{u.g.}(\Omega\cup\mathcal{C})$, with no vertices of degree greater than 4 and with $G \cap I([A]+R)$ a subgraph of $\mathbb{Z}^2$. We observe that, since $[A]$ is the top vertex of degree 4 of G inside the smaller box $[A]+R$, lemma 6 applies to G with $[A]$ and $\mathcal{C}$ as above. Let G' be the closed Eulerian graph obtained from G via lemma 6. Note that $G'\backslash\text{u.g.}(\mathcal{C})$ (with any vertices that were induced by $\mathcal{C}$ in G suppressed) is a subgraph of $\mathbb{Z}^2$ and has an open Euler trail E', with $\deg_{E'}([A]) = \deg_{G'}([A]) = 2$.

Finally, we set $\mathbb{G}' = \text{u.g.}(E')$, and show that the open trail E', obtained in either case (A) or case (B) above, gives $\mathbb{G}'$ its desired properties, as follows.

For case (A), the properties (i), (ii), (iii) and (v) follow immediately from the fact that the only moves made in order to obtain E' from $E_{\mathbb{G}}$, were to remove edges from $E_{\mathbb{G}}$, within the box $[A] + R$. Next, because p had at most one edge removed from its original set of at least 4 edges and because $\deg_{E'}([A]) = 2$, we know that $m \geqslant 4$. Additionally, because no more than 216 edges could have been removed, we know that $m \geqslant N - 216$. Putting all this together, we certainly have $\max(4, N - 220) \leqslant m \leqslant N$. Finally, the inequalities $0 \leqslant k' < K$ follow from property (v) and the fact that at least one vertex of degree 4 has had its degree reduced, namely $[A]$. Hence property (iv) is proved.

For case (B), properties (ii) and (iii) follow from the fact that all the adjustments to $\mathbb{G}$ were made inside $[A] + R$, except possibly for the removal of at most two edges to ensure $[a]$ and $[b]$ lie outside $[A] + R$. Next, property (v) follows immediately from lemma 6, since any vertices of degree 4 introduced by the addition of $\mathcal{C}$ are either removed or suppressed, once $\mathcal{C}$ is removed. Now, since property (v) is true, then as argued for case (A), $0 \leqslant k' < K$. Finally, since lemma 6 affects only the n edges of Ω, we have that the m edges of E' are bounded by $\max(12, N - 210) \leqslant m \leqslant N$, and so certainly property (iv) is satisfied, and since property (i) was shown at the end of case (B), we are done. $\square$

7. Removing all the vertices of degree 4

In this section, algorithms are developed for removing all vertices of degree 4 from an embedding of either an open or closed k-graph.

7.1. Closed k-graphs

Lemma 7. *There exists an integer $M > 0$ and a map Φ such that for all $k \geqslant 0$, for any $\tau \in \mathcal{G}_4^0(k)$, and any n-tau, σ,*

$$\sigma \overset{\Phi}{\longmapsto} (\tilde{\omega}, \Xi = (\xi_1, \xi_2, \ldots, \xi_j), \mathcal{T} = (T_1, T_2, \ldots, T_{j'})) \tag{7.1}$$

where $\tilde{\omega}$ is an m-SAP, Ξ is a j-tuple of lexicographically ordered vertices in $\mathbb{Z}^2$, and $\mathcal{T}$ is a j'-tuple of planted plane trees with non-root vertices in $\mathbb{Z}^2$, such that the following properties hold:

(1) $\max\{4, n - 4k\} \leqslant m \leqslant n$, $\lfloor \frac{k}{2} \rfloor \leqslant j \leqslant k$, *and* $\min\{1, j\} \leqslant j' \leqslant j$.

(2) $\xi_1 > \xi_2 > \cdots > \xi_j$.

(3) $\forall i$ such that $1 \leqslant i \leqslant j'$, a vertex within the tree T_i is always lexicographically smaller than any of its children in T_i, and lies within a box-distance M of any of its children. Further, the only child of the root of T_i lies on $\tilde{\omega}$.

(4) The set $\{\xi_1, \xi_2, \ldots, \xi_j\}$ is equal to the disjoint union of the non-root vertices of the j' trees in $\mathcal{T}$. Thus, if $\forall i$ m_i is the number of non-root vertices of T_i, then $\sum_{i=1}^{j'} m_i = j$.

(5) Let $R_M = \left\{(x, y) \in \mathbb{Z}^2 \middle| \frac{-M}{2} \leqslant x \leqslant \frac{M}{2}, \frac{-M}{2} \leqslant y \leqslant \frac{M}{2}\right\}$ be the box of side-length M. Then $\tilde{\omega} = \sigma$ in $\mathbb{Z}^2 \backslash \cup_{i=1}^{j} (\xi_i + R_M)$.

Proof. If $k = 0$, set $\tilde{\omega} = \sigma$, $j = j' = 0$, $\Xi = \mathcal{T} = \phi$. Otherwise, the plan is to repeatedly apply lemma 6 to the consecutive top vertices of degree 4, until all vertices of degree 4 have been removed. Because lemma 6 allows each successive move to be made inside a box of side-length 10, set $M = 10$.

Step (1). Apply lemma 6, to $G = \sigma$, with $[A] = \xi_1$, the top vertex of degree 4 of G, to obtain $\omega_1 = G'$, a closed Eulerian graph with $n_1 \leqslant n$ edges, and $k_1 < k$ vertices of degree 4.

Let $\Xi_1 = (\xi_1)$ be the 1-tuple consisting of the one vertex ξ_1, let $\mathcal{T}_1 = (T_1(\omega_1))$ be the 1-tuple consisting of one rooted plane tree with its single vertex ξ_1 being the root, and note that the root of $T_1(\omega_1)$ lies on ω_1, since $\deg_{\omega_1}(\xi_1) = 2$. (End of step (1).)

The following statement is trivially true, for $i = 1$.

Statement (i). At the end of the ith step for $i \geqslant 1$, we have $(\omega_i, \Xi_i, \mathcal{T}_i)$, where ω_i is an n_i-edge closed Eulerian graph with k_i vertices of degree 4, $\Xi_i = (\xi_1, \ldots, \xi_i)$ is an i-tuple of vertices in $\mathbb{Z}^2$, and $\mathcal{T}_i$ is an l_i-tuple, $l_i \geqslant 1$, of rooted plane trees, $\mathcal{T}_i = (T_1(\omega_i), \ldots, T_{l_i}(\omega_i))$, with vertices in $\mathbb{Z}^2$, such that the following properties hold:

(I) $n - 4i \leqslant n_i \leqslant n$ and $0 \leqslant k_i \leqslant k - i$; $1 \leqslant l_i \leqslant i$.

(II) $\xi_1 > \xi_2 > \cdots > \xi_i$.

(III) $\forall s$ such that $1 \leqslant s \leqslant l_i$, any vertex within the tree $T_s(\omega_i)$, is lexicographically smaller than its children, and is within a box-distance M of any of its children. Further, the root of each tree in $\mathcal{T}_i$ lies on ω_i.

(IV) The set $\{\xi_1, \xi_2, \ldots, \xi_i\}$ is equal to the disjoint union of the vertices of the l_i trees in $\mathcal{T}_i$. Thus, if $\forall s$ m'_s is the number of vertices of $T_s(\omega_i)$, then $\sum_{s=1}^{l_i} m'_s = i$.

(V) Let $R = \left\{(x, y) \in \mathbb{Z}^2 \middle| \frac{-M}{2} \leqslant x \leqslant \frac{M}{2}, \frac{-M}{2} \leqslant y \leqslant \frac{M}{2}\right\}$ be the box of side-length M. Then $\omega_i = \sigma$ in $S_i \equiv \mathbb{Z}^2 \backslash \cup_{s=1}^{i} (\xi_s + R)$. (End of statement ($i$).)

Next, for $i \geqslant 1$, and $k_i > 0$, perform step $(i + 1)$ below while ω_i has at least one vertex of degree 4 ($k_i > 0$). Then go to the final step.

In the description of step $(i + 1)$, statement (i) is assumed to be true and it is proved that, for $k_i > 0$, statement $(i + 1)$ holds.

Step (i + 1). First, apply lemma 6 to $G = \omega_i$ with top vertex of degree 4, $\xi_{i+1} \equiv [A] < \xi_i$, to obtain $\omega_{i+1} = G'$, a closed Eulerian graph with $n_{i+1} \leqslant n_i$ edges and $k_{i+1} \leqslant k_i - 1$ vertices of degree 4. Because statement (i) is true, and since at each step the total number of edges may decrease by no more than 4, we have that $n - 4(i + 1) \leqslant n_i - 4 \leqslant n_{i+1} \leqslant n$ and $0 \leqslant k_{i+1} \leqslant k - (i + 1)$. Next observe that, by lemma 6, we have $\omega_{i+1} = \omega_i$ on $\mathbb{Z}^2 \backslash (\xi_{i+1} + R) \supseteq S_{i+1}$, and by property (V) of statement (i), we have $\omega_i = \sigma$ on $S_i \supseteq S_{i+1}$. Therefore, $\omega_{i+1} = \sigma$ on S_{i+1}. Thus, defining $\Xi_{i+1} \equiv \Xi_i \cup \{\xi_{i+1}\}$ (ordered from largest to smallest vertex), we have ω_{i+1} and Ξ_{i+1} which satisfy properties (II) and (V) of statement $(i + 1)$.

We next construct $\mathcal{T}_{i+1}$, from the set of trees $\mathcal{T}_i$ and the vertex ξ_{i+1}. Since we want the root of each tree in $\mathcal{T}_{i+1}$ to lie on ω_{i+1}, and since we know that ξ_{i+1} does lie on ω_{i+1} $\left(\deg_{\omega_{i+1}}(\xi_{i+1}) = 2\right)$, we must join the vertex ξ_{i+1} to every tree of $\mathcal{T}_i$, whose root may have been affected by step $(i+1)$. Specifically, we are concerned with any tree $T \in \mathcal{T}_i$ such that its root is contained in the box $\xi_{i+1} + R$, namely the set of trees $\mathcal{T}'_i = \{T \in \mathcal{T}_i \mid \text{root}(T) \in \xi_{i+1} + R\} \subseteq \mathcal{T}_i$. We form a new tree, $T_{\text{new}}(i + 1)$, by joining every tree $T \in \mathcal{T}'_i$ to ξ_{i+1} by means of a line segment in $\mathbb{R}^2$, which connects root(T) to ξ_{i+1}, and set $\text{root}(T_{\text{new}}(i + 1)) \equiv \xi_{i+1}$. Of course, it is possible that $\mathcal{T}'_i = \emptyset$, in which case, simply set $T_{\text{new}}(i + 1)$ to the rooted plane tree with single vertex ξ_{i+1}.

Note that, by the above construction, for each $v \in T_{\text{new}}(i + 1) \backslash \xi_{i+1}$ there exists an s such that v and all of its offspring lie in $T_s(\omega_i)$. Also, $\text{root}(T_{\text{new}}(i+1)) = \xi_{i+1} < v$. Therefore, every vertex in $T_{\text{new}}(i+1)$ is lexicographically smaller than its children, and lies within a box-distance M of each of its children. Also by construction, $\text{root}(T_{\text{new}}(i + 1))$ lies on ω_{i+1}. Therefore, set $\mathcal{T}_{i+1} \equiv (\mathcal{T}_i \backslash \mathcal{T}'_i) \cup \{T_{\text{new}}(i + 1)\}$ and $T_{l_{i+1}}(\omega_{i+1}) = T_{\text{new}}(i + 1)$. Since $0 \leqslant \text{card}(\mathcal{T}_i \backslash \mathcal{T}'_i) \leqslant l_i \leqslant i$, it follows that $1 \leqslant \text{card}(\mathcal{T}_{i+1}) \equiv l_{i+1} \leqslant i + 1$. Thus, the elements of $\mathcal{T}_{i+1}$ may be relabelled (preserving the previous order of the trees in $\mathcal{T}_i \backslash \mathcal{T}'_i$) to give $\mathcal{T}_{i+1} \equiv (T_1(\omega_{i+1}), \ldots, T_{l_{i+1}}(\omega_{i+1}))$. We have thus constructed $\mathcal{T}_{i+1}$ which satisfies properties (I) and (III) of statement $(i + 1)$, since the root of any tree in $\mathcal{T}_i \backslash \mathcal{T}'_i$ is unaffected by the moves made in step $(i + 1)$.

Now we only have to verify property (IV) of statement $(i+1)$. It is clear from the construction of $\mathcal{T}_{i+1}$ that the set $\{\xi_1, \ldots, \xi_{i+1}\}$ is equal to the vertex set of the disjoint union of the l_{i+1} trees in $\mathcal{T}_{i+1}$, so that $\sum_{s=1}^{l_{i+1}} m'_s = i+1$, where $\forall s$, m'_s is the number of vertices in $T_s(\omega_{i+1})$. Thus, statement $(i+1)$ is proved. (End of step $(i+1)$.)

Final step. Since σ had a finite number of vertices of degree 4, $\exists\, j \geqslant 1$, so that we have performed step (i) and verified statement (i), for $1 \leqslant i \leqslant j$, and ω_j has $k_j = 0$. Hence, either $j = 1$, or else $k > k_1 > k_2 > \cdots > k_{j-1} > k_j = 0$. We thus define the triple $(\tilde{\omega}, \Xi, \mathcal{T})$ that satisfies the lemma, as follows: set $m \equiv n_j$ and $\tilde{\omega} \equiv \omega_j$, which is an m-edge polygon, since it is an m-edge closed Eulerian graph with $k_j = 0$ vertices of degree 4, and we set $\Xi \equiv \Xi_j$, $j' \equiv l_j$. Lastly, we construct the set of planted plane trees, $\mathcal{T}$, from the set of rooted plane trees, $\mathcal{T}_j$, as follows. Note that by the above construction, for any $1 \leqslant s < t \leqslant j'$ with $\mathrm{root}(T_s(\omega_j)) = \xi_{s_j}$ and $\mathrm{root}(T_t(\omega_j)) = \xi_{t_j}$, $\xi_{s_j} \notin (\xi_{t_j} + R)$. So, for each $1 \leqslant s \leqslant j'$, the planted plane tree T_s is obtained from $T_s(\omega_j)$, by attaching the plant edge $\{\xi_{s_j}, \xi_{s_j} - u_1/2 - u_2/2\}$ and the new unique root vertex $\xi_{s_j} - u_1/2 - u_2/2$ to $T_s(\omega_j)$. Then, set $\mathcal{T} \equiv (T_1, T_2, \ldots, T_{j'})$.

We observe that $j \leqslant k$, since no more than k steps are required to remove k vertices of degree 4 from σ. Also, since lemma 6 removes no more than two vertices of degree 4 at a time, the removal of k vertices of degree 4 cannot be accomplished in fewer than $\lfloor \frac{k}{2} \rfloor$ steps. Hence, $\lfloor \frac{k}{2} \rfloor \leqslant j \leqslant k$. The rest of the lemma is an immediate consequence of the fact that statement (j) is true. □

7.2. Open k-graphs

Let $\mathbb{G}$ be an n-edge embedding in $\mathbb{Z}^2$ of a graph in $\mathcal{G}_4^2(0)$. For brevity, we refer to such an embedding as an n-USAWFF, since it can be made into an undirected SAW (USAW) by deleting the first and/or final step of an Euler trail of the embedding.

We now pose the analogue of lemma 7 for open k-graphs. We leave it to the reader to show that the following lemma may be proved, *mutatis mutandis*, using corollary 6 in the same way as lemma 6 is used to prove lemma 7.

Lemma 8. *There exists an integer $\mathcal{M} > 0$ and a map Ψ, such that for all $k \geqslant 1$, for any $\tau \in \mathcal{G}_4^2(k)$, and any n-tau, ρ,*

$$\rho \overset{\Psi}{\longmapsto} (\tilde{\omega}, \Xi = (\xi_1, \xi_2, \ldots, \xi_j), \mathcal{T} = (T_1, T_2, \ldots, T_{j'})) \tag{7.2}$$

where $\tilde{\omega}$ is an m-USAWFF, Ξ is a j-tuple of lexicographically ordered vertices in $\mathbb{Z}^2$, and $\mathcal{T}$ is a j'-tuple of planted plane trees with non-root vertices in $\mathbb{Z}^2$, such that the following properties hold:

(1) $\max\{4, n - 220k\} \leqslant m \leqslant n$, $0 \leqslant j \leqslant k$, *and* $\min\{1, j\} \leqslant j' \leqslant j$.

(2) $\xi_1 > \xi_2 > \cdots > \xi_j$.

(3) $\forall i$ such that $1 \leqslant i \leqslant j'$, any vertex in the tree T_i, is lexicographically smaller than its children, and lies within a box-distance $\mathcal{M}$ of any of its children. Further, the only child of the root of T_i is an even vertex of $\tilde{\omega}$.

(4) The set $\{\xi_1, \xi_2, \ldots, \xi_j\}$ is equal to the disjoint union of the non-root vertices of the j' trees in $\mathcal{T}$. Thus, if $\forall i$ m_i is the number of non-root vertices of T_i, then $\sum_{i=1}^{j'} m_i = j$.

(5) Let $R_{\mathcal{M}} = \left\{(x, y) \in \mathbb{Z}^2 \middle| \frac{-\mathcal{M}}{2} \leqslant x \leqslant \frac{\mathcal{M}}{2}, \frac{-\mathcal{M}}{2} \leqslant y \leqslant \frac{\mathcal{M}}{2}\right\}$ be the box of side-length $\mathcal{M}$. Then $\tilde{\omega} = \rho$ in $\mathbb{Z}^2 \backslash \cup_{i=1}^{j} (\xi_i + R_{\mathcal{M}})$.

While in section 7.1, the goal was to relate embeddings of closed k-graphs to polygons, the goal in this section is to relate embeddings of open k-graphs to USAWs. Lemma 8 almost gets us there, by relating embeddings of open k-graphs to USAWFFs. We need to take one more step, by relating the set $\mathcal{U}$ of m-USAWFFs to the set $\mathcal{V}$ of $(m-2)$-USAWs. We do this by assigning the map $\Upsilon : \mathcal{U} \to \mathcal{V}$, which transforms any m-USAWFF, $G \in \mathcal{U}$, into an $(m-2)$-USAW, $W \in \mathcal{V}$, by removing two edges from G in the following manner. Fix an Euler trail, E_G of G, and label the first and last directed edges as α and β. Then

$$W = \Upsilon(G) \equiv \text{u.g.}(E_G \backslash \alpha \backslash \beta). \tag{7.3}$$

Since there are $3 \times 3 = 9$ ways to add one edge to each of the degree-1 vertices of a USAW, the following observation is true, concerning the map Υ.

Lemma 9. *For graphs in $\mathbb{Z}^2$, the pre-image of Υ with largest cardinality has no more than nine members: $\max_{F \in \mathcal{V}} |\Upsilon^{-1}(F)| \leqslant 9$.*

8. Main theorem upper bounds

8.1. Closed k-graphs

Lemma 10. *There exists a constant $C > 1$, such that for all $k \geqslant 0$*

$$\overset{\circ}{E}_n(k) \leqslant C^k \binom{2n}{k} p_n. \tag{8.1}$$

Proof. Let $S_{n,k}$ be the set of all n-edge embeddings of closed k-graphs in $\mathbb{Z}^2$ which have their top vertex of degree 4 located at the origin (if $k = 0$, then the top vertex of the embedding is located at the origin). For a given n and k, let $L_{n,k} = |\Phi(S_{n,k})|$ be the number of distinct images under Φ of elements of $S_{n,k}$, where $\Phi(S_{n,k}) = \{(\omega^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})\}_{1 \leqslant i \leqslant L_{n,k}}$. Here, i is a member of the index set of all distinct images under Φ, and we emphasize that its value is *not* being used to denote the size of the polygon, vertex list or tree list in the triple $(\omega^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})$. Then

$$\overset{\circ}{E}_n(k) = \sum_{\sigma \in S_{n,k}} 1 = \sum_{i=1}^{L_{n,k}} |\Phi^{-1}(\omega^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})| \leqslant L_{n,k} \max_i |\Phi^{-1}(\omega^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})|. \tag{8.2}$$

We shall calculate two upper bounds $N_1(k)$ and $N_2(n,k)$, with $\max_i |\Phi^{-1}(\omega^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})| \leqslant N_1(k)$ and $L_{n,k} \leqslant N_2(n,k)$, and there will be a C such that $N_1(k) N_2(n,k) = C^k \binom{2n}{k} p_n$, which will give us the desired result.

Given $j_i \equiv |\Xi^{(i)}|$, note that the number of ways to add or delete edges in j_i boxes is bounded above by $(2^E)^{j_i}$, where $E = 2M(M+1)$ is the number of edges in an $M \times M$ box. Since the vertices specified by $\Xi^{(i)} = (\xi_1, \xi_2, \ldots, \xi_{j_i})$ determine the centres of the j_i boxes in which changes to a $\sigma \in \Phi^{-1}(\omega^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})$ were made, we have

$$\max_i |\Phi^{-1}(\omega^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})| \leqslant \max_i (2^E)^{j_i} \leqslant \left(2^{4M^2}\right)^k \equiv N_1(k) \tag{8.3}$$

where the last inequality follows from the fact that $j_i \leqslant k$ and $E \leqslant 4M^2$.

In order to calculate $N_2(n,k)$, we first note that

$$L_{n,k} \leqslant \sum_{m=n-4k}^{n} \sum_{j=\lfloor \frac{k}{2} \rfloor}^{k} (\text{\# ways to form a triple } (\omega_m, \Xi_j, \mathcal{T}_{j'})) \tag{8.4}$$

where ω_m is an m-edge polygon, Ξ_j is a lexicographically ordered j-tuple of vertices in $\mathbb{Z}^2$, and $\mathcal{T}_{j'} = (T_1, T_2, \ldots, T_{j'})$ is a j'-tuple of planted plane trees. This is an inequality because the sum is taken over all possible m-edge polygons, regardless of whether $(\omega_m, \Xi_j, \mathcal{T}_{j'}) \in \Phi(S_{n,k})$ or not. Letting m_i be the number of non-root vertices of T_i we have $\sum_{i=1}^{j'} m_i = j$.

The number, P_l, of abstract planted plane trees, with $l \geqslant 1$ non-root vertices is given by [22]

$$P_l = \frac{1}{l}\binom{2(l-1)}{l-1} \tag{8.5}$$

and

$$P_{l_1} P_{l_2} \leqslant P_{l_1+l_2-1}. \tag{8.6}$$

From equation (8.5), we also note that

$$P_l \leqslant \binom{2(l-1)}{l-1} \leqslant 2^{2(l-1)} = 4^{(l-1)}. \tag{8.7}$$

Because of property (3) in lemma 7, once the child of the root of T_i is chosen from the vertices of ω_m, the number of ways to choose the remaining $m_i - 1$ non-root vertices is bounded by V^{m_i-1}, where $V = 2M(M+1)$ is the maximum number of ways to choose a child vertex within a box-distance M from its parent vertex. Thus, once we have chosen j' sites on ω_m and a j'-tuple $(m_1, m_2, \ldots, m_{j'})$, an upper bound on the number of possible j'-tuples $(T_1, T_2, \ldots, T_{j'})$ is given by $\prod_{i=1}^{j'} P_{m_i} V^{m_i-1}$, and consequently, for $k \geqslant 1$

$$L_{n,k} \leqslant \sum_{m=n-4k}^{n} \sum_{j=\lfloor\frac{k}{2}\rfloor}^{k} p_m \sum_{j'=1}^{j} \binom{m}{j'} \sum_{\{m_i\}} \prod_{i=1}^{j'} \left(P_{m_i} V^{m_i-1}\right) \tag{8.8}$$

where $\sum_{\{m_i\}}$ denotes the sum over $\left\{m_i \geqslant 1, 1 \leqslant i \leqslant j' \,\middle|\, \sum_{i=1}^{j'} m_i = j\right\}$. We note also that for $j \geqslant 1$ and $1 \leqslant j' \leqslant j$

$$\sum_{\{m_i\}} 1 = \binom{j-1}{j'-1} = \binom{j-1}{j-j'}. \tag{8.9}$$

Next, because $p_m \leqslant p_n$ and $V \leqslant 4M^2$, and by applying equations (8.6), (8.7) and (8.9) to (8.8), we have for $k \geqslant 1$

$$\begin{aligned} L_{n,k} &\leqslant p_n \sum_{m=n-4k}^{n} \sum_{j=\lfloor\frac{k}{2}\rfloor}^{k} \sum_{j'=1}^{j} \binom{m}{j'} \sum_{\{m_i\}} P_{j-j'+1} V^{j-j'} \\ &\leqslant p_n \sum_{m=n-4k}^{n} \sum_{j=\lfloor\frac{k}{2}\rfloor}^{k} \sum_{j'=1}^{j} \binom{m}{j'} 4^{j-j'} (4M^2)^{j-j'} \sum_{\{m_i\}} 1 \\ &\leqslant p_n 4^k (4M^2)^k \sum_{m=n-4k}^{n} \sum_{j=\lfloor\frac{k}{2}\rfloor}^{k} \sum_{j'=1}^{j} \binom{m}{j'} \binom{j-1}{j-j'} \\ &= p_n (16M^2)^k \sum_{m=n-4k}^{n} \sum_{j=\lfloor\frac{k}{2}\rfloor}^{k} \sum_{j'=1}^{j} \binom{m}{j'} \binom{j-1}{j-j'}. \end{aligned} \tag{8.10}$$

We apply the Vandermonde convolution identity to the innermost sum on the last line of (8.10) and use the fact that $m + j - 1 \leqslant n + n - 1 \leqslant 2n$ to obtain

$$\sum_{j'=1}^{j} \binom{m}{j'} \binom{j-1}{j-j'} = \binom{m+j-1}{j} \leqslant \binom{2n}{j} \leqslant \binom{2n}{k}. \tag{8.11}$$

The last inequality in (8.11) is true since $0 \leqslant j \leqslant k \leqslant n/2$ and $\binom{2n}{l}$ is an increasing function of l on the interval $0 \leqslant l \leqslant n$.

Next, by using equation (8.11) and the fact that $(4k+1)(k+1) \leqslant 10^k, \forall k \geqslant 1$, equation (8.10) becomes for all $k \geqslant 1$

$$\begin{aligned} L_{n,k} &\leqslant p_n (16M^2)^k \sum_{m=n-4k}^{n} \sum_{j=\lfloor \frac{k}{2} \rfloor}^{k} \binom{2n}{k} \\ &\leqslant p_n (16M^2)^k (4k+1)(k+1) \binom{2n}{k} \\ &\leqslant p_n (16M^2)^k 10^k \binom{2n}{k} \\ &\leqslant p_n (2^8 M^2)^k \binom{2n}{k} \equiv N_2(n,k). \end{aligned} \tag{8.12}$$

Finally, putting equations (8.2), (8.3) and (8.12) together, we have for all $k \geqslant 1$

$$\begin{aligned} \mathring{E}_n(k) &\leqslant N_2(n,k) N_1(k) \\ &= p_n (2^8 M^2)^k \binom{2n}{k} \left(2^{4M^2}\right)^k = \left(2^{8+4M^2} M^2\right)^k \binom{2n}{k} p_n. \end{aligned} \tag{8.13}$$

Noting that the final upper bound in equation (8.13) is also true for $k = 0$, we have obtained the required result, provided $C \geqslant 2^{8+4M^2} M^2$. □

8.2. Open k-graphs

Lemma 11. *There exist constants $D_0 > 0$ and $D > 1$, such that for all $k \geqslant 0$*

$$\check{E}_n(k) \leqslant D_0 (D)^k \binom{2n}{k} c_n. \tag{8.14}$$

Proof. For ρ an n-tau, with $\tau \in \mathcal{G}_4^2(k)$ and $k \geqslant 1$, define the composition mapping $\Upsilon \circ \Psi$

$$\rho \xrightarrow{\Psi} (\tilde{\omega}, \Xi, \mathcal{T}) \xrightarrow{\Upsilon} (\tilde{\sigma}, \Xi, \mathcal{T}) \tag{8.15}$$

where $(\tilde{\omega}, \Xi, \mathcal{T}) = \Psi(\rho)$ as defined in lemma 8, and $\tilde{\sigma} = \Upsilon(\tilde{\omega})$ as defined by equation (7.3). We note that any tree $T \in \mathcal{T}$ will remain attached (by the only child of its root) to $\tilde{\sigma}$ under the mapping Υ, because of property (3) of lemma 8. Hence, the composition $\Upsilon \circ \Psi$ is well defined.

Let $P_{n,k}$ be the set of all n-edge embeddings of open k-graphs in $\mathbb{Z}^2$ which have their top vertex of degree 4 located at the origin (if $k = 0$, then the top odd vertex of the embedding is located at the origin). For a given n and $k \geqslant 1$, the number of distinct images, $\tilde{L}_{n,k}$, under $\Upsilon \circ \Psi$ of elements of $P_{n,k}$, is given by $\tilde{L}_{n,k} = |\Upsilon \circ \Psi(P_{n,k})|$, where $\Upsilon \circ \Psi(P_{n,k}) = \{\tilde{\sigma}^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)}\}_{1 \leqslant i \leqslant \tilde{L}_{n,k}}$. Then, as was done for closed k-graphs in section 8.1, we obtain

$$\check{E}_n(k) = \sum_{\rho \in P_{n,k}} 1 = \sum_{i=1}^{\tilde{L}_{n,k}} |(\Upsilon \circ \Phi)^{-1}(\tilde{\sigma}^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})| \leqslant \tilde{L}_{n,k} \max_i |(\Upsilon \circ \Phi)^{-1}(\tilde{\sigma}^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})|. \tag{8.16}$$

Using arguments analogous to those used in proving lemma 10, we obtain the following upper bound:

$$\max_i |(\Upsilon \circ \Phi)^{-1}(\tilde{\sigma}^{(i)}, \Xi^{(i)}, \mathcal{T}^{(i)})| \leqslant 9\left(2^{4\mathcal{M}^2}\right)^k \equiv \tilde{N}_1(k) \tag{8.17}$$

where $\mathcal{M}$ is as given in lemma 8. The extra factor of 9 that appears in the inequality above, comes from invoking lemma 9.

The upper bound $\tilde{N}_2(n, k) \geqslant \tilde{L}_{n,k}$ is also obtained using arguments analogous to those given in the proof of lemma 10. Here, we set $\tilde{V} = 2\mathcal{M}(\mathcal{M}+1)$, and observe that a USAW (an embedding of τ_0^2) with m edges has $m-1$ even vertices and $g_m(\tau_0^2) = c_m/2$. We also take care to note that, since k is not more than half the number of even vertices in any n-edged ρ, we have $1 \leqslant j \leqslant k \leqslant (n-1)/2$, and hence the inequality $\binom{2n}{j} \leqslant \binom{2n}{k}$ still stands. Thus, our calculation proceeds as follows for $k \geqslant 1$:

$$\begin{aligned}
\tilde{L}_{n,k} &\leqslant \sum_{m=n-220k-2}^{n} \frac{c_m}{2} \sum_{j=1}^{k} \sum_{j'=1}^{j} \binom{m-1}{j'} \sum_{\{m_i\}} \prod_{i=1}^{j'} \left(P_{m_i} \tilde{V}^{m_i - 1}\right) \\
&\leqslant \frac{c_n}{2} \sum_{m=n-220k-2}^{n} \sum_{j=1}^{k} \sum_{j'=1}^{j} \binom{m}{j'} P_{j-j'+1} \tilde{V}^{j-j'} \sum_{\{m_i\}} 1 \\
&\leqslant \frac{c_n}{2} \sum_{m=n-220k-2}^{n} \sum_{j=1}^{k} \sum_{j'=1}^{j} 4^{j-j'} \tilde{V}^{j-j'} \binom{m}{j'} \binom{j-1}{j-j'} \\
&\leqslant \frac{c_n}{2} 4^k \tilde{V}^k \sum_{m=n-220k-2}^{n} \sum_{j=1}^{k} \binom{m+j-1}{j} \\
&\leqslant \frac{c_n}{2} 4^k (4\mathcal{M}^2)^k \sum_{m=n-220k-2}^{n} k \binom{2n}{k} \\
&\leqslant \frac{c_n}{2} \binom{2n}{k} (2^4 \mathcal{M}^2)^k k(220k+3) \\
&\leqslant \frac{c_n}{2} \binom{2n}{k} (2^{12} \mathcal{M}^2)^k \equiv \tilde{N}_2(n, k).
\end{aligned} \tag{8.18}$$

The last inequality in equation (8.18) is obtained by noting that $k(220k+3) \leqslant 2^{8k}, \forall k \geqslant 1$. Finally, putting equations (8.16)–(8.18) together, we have for all $k \geqslant 1$

$$\begin{aligned}
\breve{E}_n(k) &\leqslant \tilde{N}_2(n, k) \tilde{N}_1(k) \\
&= \frac{c_n}{2} \binom{2n}{k} (2^{12} \mathcal{M}^2)^k \times 9\left(2^{4\mathcal{M}^2}\right)^k = \frac{9}{2} \left(2^{12+4\mathcal{M}^2} \mathcal{M}^2\right)^k \binom{2n}{k} c_n.
\end{aligned} \tag{8.19}$$

Note that the final upper bound in equation (8.19) is also true for $k = 0$. Hence the desired result is obtained provided that we choose $D \geqslant 2^{12+4\mathcal{M}^2} \mathcal{M}^2$ and $D_0 = \frac{9}{2}$. □

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